

Coefficient bounds for certain subclasses of bi-prestarlike functions associated with the Chebyshev polynomials

H. Ö. GÜNEY, G. MURUGUSUNDARAMOORTHY,
K. VIJAYA, K. THILAGAVATHI

ABSTRACT. In this paper, we introduce and investigate a new subclass of bi-prestarlike functions defined in the open unit disk, associated with Chebyshev Polynomials. Furthermore, we find estimates of first two coefficients of functions in these classes, making use of the Chebyshev polynomials. Also, we obtain the Fekete-Szegő inequalities for function in these classes. Several consequences of the results are also pointed out as corollaries.

1. INTRODUCTION

Let \mathcal{A} denote the class of analytic functions of the form

$$(1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

normalized by the conditions $f(0) = 0 = f'(0) - 1$ defined in the open unit disk

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}.$$

Let \mathcal{S} be the subclass of \mathcal{A} consisting of functions of the form (1) which are also univalent in Δ .

An analytic function φ is subordinate to an analytic function ψ , written $\varphi(z) \prec \psi(z)$, provided there is an analytic function ω defined on Δ with $\omega(0) = 0$ and $|\omega(z)| < 1$ satisfying $\varphi(z) = \psi(\omega(z))$.

Let $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the well-known subclasses of \mathcal{S} , consisting of starlike and convex functions of order α , $0 \leq \alpha < 1$, respectively.

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The function

$$s(z) = \frac{z}{(1-z)^{2(1-\alpha)}} = z + \sum_{n=2}^{\infty} \Psi_n(\alpha) z^n,$$

where

$$(2) \quad \Psi_n(\alpha) = \left(\frac{\prod_{k=2}^n (k-2\alpha)}{(n-1)!} \right)$$

is the well-known extremal function for the class $\mathcal{S}^*(\alpha)$. Also $f \in \mathcal{A}$ is said to be prestarlike functions of order α ($0 \leq \alpha < 1$), denoted by $\mathcal{R}(\alpha)$ if $f * s(z) \in \mathcal{S}^*(\alpha)$. We note that $\mathcal{R}(1/2) = \mathcal{S}^*(1/2)$ and $\mathcal{R}(0) = \mathcal{K}(0)$. Using the convolution techniques, Ruscheweyh [18] introduced and studied the class of prestarlike functions of order α .

For functions $f \in \mathcal{S}$, we have

$$f \in \mathcal{K}(0) \iff z f' \in \mathcal{S}^*(0).$$

The Koebe one quarter theorem [5] ensures that the image of Δ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \quad z \in \Delta.$$

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}).$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ . Let Σ denote the class of bi-univalent functions defined in the unit disk Δ . Since $f \in \Sigma$ has the Maclaurian series given by (1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$(3) \quad g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 + \dots$$

We notice that the class Σ is not empty. For example, the functions z , $\frac{z}{1-z}$, $-\log(1-z)$ and $\frac{1}{2} \log \frac{1+z}{1-z}$ are members of Σ . However, the Koebe function is not a member of Σ . In fact, Srivastava et al. [19] have actually revived the study of analytic and bi-univalent functions in recent years, it was followed by such works as those by (see [2, 3, 7, 11, 12, 14, 16, 17, 19, 31, 32, 33, 34]).

In Geometric Function Theory, there have been many interesting and fruitful usages of a wide variety of special functions, q -calculus and special polynomials (for example) the Fibonacci polynomials, Faber polynomials the Lucas polynomials, the Pell polynomials, the Pell-Lucas polynomials, and the Chebyshev polynomials of the second kind and Horadam polynomials are potentially important in a variety of disciplines in the mathematical, physical, statistical, and engineering sciences. These polynomials have been studied in several papers from a theoretical point of view. Lately, there has been triggering interest by introducing a new class of \mathcal{A}

and discussed coefficient problems and celebrated Fekete-Szegő problem (see [21, 26, 27, 28]) further certain subclasses of Σ were defined by means of these polynomials (see [15, 20, 22, 24] and discussed extensively. Here, in this article, we propose to make use of the Chebyshev polynomials, which is used by us in this paper, play a considerable act in numerical analysis. We know that the Chebyshev polynomials are four kinds. The most of books and research articles related to specific orthogonal polynomials of Chebyshev family, contain essentially results of Chebyshev polynomials of first and second kinds $T_n(x)$ and $U_n(x)$ and their numerous uses in different applications, see Doha [4], Dziok et al. [6] and Mason [13].

The well-known kinds of the Chebyshev polynomials are the first and second kinds. In the case of real variable x on $(-1, 1)$, the first and second kinds are defined by

$$T_n(x) = \cos n\theta,$$

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin \theta},$$

where the subscript n denotes the polynomial degree and where $x = \cos \theta$. We consider the function

$$\Phi(z, t) = \frac{1}{1 - 2tz + z^2}.$$

We note that if $t = \cos \tau$, $\tau \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, then for all $z \in \Delta$

$$\begin{aligned} \Phi(z, t) &= \frac{1}{1 - 2tz + z^2} = 1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\tau}{\sin \tau} z^n \\ &= 1 + 2 \cos \tau z + (3 \cos^2 \tau - \sin^2 \tau) z^2 + \dots \end{aligned}$$

Thus, we write

$$\Phi(z, t) = 1 + U_1(t)z + U_2(t)z^2 + \dots, \quad z \in \Delta, t \in (-1, 1),$$

where $U_{n-1} = \frac{\sin(n \arccos t)}{\sqrt{1-t^2}}$ for $n \in \mathbb{N}$, are the second kind of the Chebyshev polynomials. Also, it is known that

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t),$$

and

$$(4) \quad U_1(t) = 2t, \quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \dots$$

The Chebyshev polynomials $T_n(t)$, $t \in [-1, 1]$, of the first kind have the generating function of the form

$$\sum_{n=0}^{\infty} T_n(t)z^n = \frac{1 - tz}{1 - 2tz + z^2}, \quad z \in \Delta.$$

All the same, the Chebyshev polynomials of the first kind $T_n(t)$ and the second kind $U_n(t)$ are well connected by the following relationship

$$\begin{aligned} \frac{dT_n(t)}{dt} &= nU_{n-1}(t), \\ T_n(t) &= U_n(t) - tU_{n-1}(t), \\ 2T_n(t) &= U_n(t) - U_{n-2}(t). \end{aligned}$$

Several authors have introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients (see [2, 3, 11, 12, 19, 23, 31]), bi-close-to-convex functions[8, 10] and m - fold symmetric functions by [25, 30]. Recently, Jahangiri and Hamidi [9] introduced and studied certain subclasses of bi-prestarlike functions mentioned as below:

The expansion of $s(z) = \frac{z}{(1-z)^{2(1-\alpha)}}$ is given by

$$\begin{aligned} s(z) &= z + \frac{(2-2\alpha)}{1!}z^2 + \frac{(2-2\alpha)(3-2\alpha)}{2!}z^3 \\ &+ \frac{(2-2\alpha)(3-2\alpha)(4-2\alpha)}{3!}z^4 + \dots . \end{aligned}$$

So, by the definition of Hadamard product, we have

$$\begin{aligned} F(z) &= \frac{z}{(1-z)^{2(1-\alpha)}} * f(z) \\ &= s(z) * f(z) \\ &= z + \frac{(2-2\alpha)a_2}{1!}z^2 + \frac{(2-2\alpha)(3-2\alpha)a_3}{2!}z^3 \\ &+ \frac{(2-2\alpha)(3-2\alpha)(4-2\alpha)a_4}{3!}z^4 + \dots \end{aligned}$$

equivalently

$$(5) \quad F(z) = z + \Psi_2(\alpha)a_2z^2 + \Psi_3(\alpha)a_3z^3 + \Psi_4(\alpha)a_4z^4 + \dots .$$

Similarly, for the inverse function $g = f^{-1}$, we note that $G(w) = F^{-1}(z)$ and is obtained as

$$G(w) = w - \frac{(2-2\alpha)a_2}{1!}w^2 + \frac{\left(4(2-2\alpha)^2a_2^2 - (2-2\alpha)(3-2\alpha)a_3\right)}{2!}w^3 + \dots$$

equivalently

$$(6) \quad G(w) = w - \Psi_2(\alpha)a_2w^2 + (2\Psi_2^2(\alpha)a_2^2 - \Psi_3(\alpha)a_3)w^3 + \dots .$$

In this paper, motivated by recent works of Altınkaya and Yalçın [1] and Jahangiri and Hamidi [9], we introduce a subclass bi-prestarlike function class associated with Chebyshev polynomials and obtain the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the functions $f \in \mathcal{R}_\Sigma(\lambda, \alpha, \Phi)$ by subordination and consider the celebrated Fekete-Szegö problem. We also provide relevant connections of our results with those considered in earlier investigations.

Now we define a new subclass bi-prestarlike functions in the open unit disk, associated with Chebyshev Polynomials as below.

Definition 1. For $0 \leq \lambda \leq 1$, and $t \in (0, 1)$ a function $f \in \Sigma$ of the form (1) is said to be in the class $\mathcal{R}_\Sigma(\lambda, \alpha, \Phi)$ if the following subordination holds:

$$(7) \quad (1 - \lambda) \frac{zF'(z)}{F(z)} + \lambda \left(1 + \frac{zF''(z)}{F'(z)} \right) \prec \Phi(z, t)$$

and

$$(8) \quad (1 - \lambda) \frac{wG'(w)}{G(w)} + \lambda \left(1 + \frac{wG''(w)}{G'(w)} \right) \prec \Phi(w, t),$$

where $z, w \in \Delta$ and F and G is given by (5) and (6), respectively.

Remark 1. Suppose $f \in \Sigma$. Then $\mathcal{R}_\Sigma(0, \alpha, \Phi(z, t)) \equiv \mathcal{PS}_\Sigma^*(\alpha, \Phi)$: thus $f \in \mathcal{PS}_\Sigma^*(\alpha, \Phi)$ if the following subordination holds:

$$\frac{zF'(z)}{F(z)} \prec \Phi(z, t) \quad \text{and} \quad \frac{wG'(w)}{G(w)} \prec \Phi(w, t),$$

where $z, w \in \Delta$ and G is given by (6).

Remark 2. Suppose $f \in \Sigma$. Then $\mathcal{R}_\Sigma(1, \alpha,) \equiv \mathcal{K}_\Sigma^*(\alpha, \Phi)$: thus $f \in \mathcal{K}_\Sigma^*(\alpha, \Phi)$ if the following subordination holds:

$$1 + \frac{zF''(z)}{F'(z)} \prec \Phi(z, t) \quad \text{and} \quad 1 + \frac{wG''(w)}{G'(w)} \prec \Phi(w, t),$$

where $z, w \in \Delta$ and g is given by (6).

2. INITIAL TAYLOR COEFFICIENTS $f \in \mathcal{R}_\Sigma(\lambda, \alpha, \Phi)$

Theorem 1. Let f given by (1) be in the class $\mathcal{R}_\Sigma(\lambda, \alpha, \Phi)$ and $t \in (0, 1)$. Then

$$(9) \quad |a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{[2(1 + 2\lambda)\Psi_3(\alpha) - (\lambda^2 + 5\lambda + 2)\Psi_2^2(\alpha)]4t^2 + (1 + \lambda)^2\Psi_2^2(\alpha)}}$$

and

$$(10) \quad |a_3| \leq \frac{4t^2}{(1 + \lambda)^2\Psi_2^2(\alpha)} + \frac{t}{(1 + 2\lambda)\Psi_3(\alpha)},$$

where $0 \leq \lambda \leq 1$ and $t \neq \frac{(1+\lambda)\Psi_2(\alpha)}{\sqrt{(\lambda^2+5\lambda+2)\Psi_2^2(\alpha)-2(1+2\lambda)\Psi_3(\alpha)}}$.

Proof. Let $f \in \mathcal{R}_\Sigma(\lambda, \alpha, \Phi)$ and $g = f^{-1}$. Considering (7) and (8), we have

$$(11) \quad (1 - \lambda) \frac{zF'(z)}{F(z)} + \lambda \left(1 + \frac{zF''(z)}{F'(z)} \right) = \Phi(z, t)$$

and

$$(12) \quad (1 - \lambda) \frac{wG'(w)}{G(w)} + \lambda \left(1 + \frac{wG''(w)}{G'(w)} \right) = \Phi(w, t).$$

Define the functions $u(z)$ and $v(w)$ by

$$(13) \quad u(z) = c_1z + c_2z^2 + \dots$$

and

$$(14) \quad v(w) = d_1w + d_2w^2 + \dots$$

are analytic in Δ with $u(0) = 0 = v(0)$ and $|u(z)| < 1, |v(w)| < 1$, for all $z, w \in \Delta$. It is well-known that

$$(15) \quad \begin{aligned} |u(z)| &= |c_1z + c_2z^2 + \dots| < 1, \\ |v(w)| &= |d_1w + d_2w^2 + \dots| < 1, z, w \in \Delta, \end{aligned}$$

then

$$(16) \quad |c_j| \leq 1, \quad |d_j| \leq 1, \quad \text{for all } j \in \mathbb{N}.$$

Using (13) and (14) in (11) and (12) respectively, we have

$$(17) \quad (1 - \lambda) \frac{zF'(z)}{F(z)} + \lambda \left(1 + \frac{zF''(z)}{F'(z)} \right) = 1 + U_1(t)u(z) + U_2(t)u^2(z) + \dots,$$

and

$$(18) \quad (1 - \lambda) \frac{wG'(w)}{G(w)} + \lambda \left(1 + \frac{wG''(w)}{G'(w)} \right) = 1 + U_1(t)v(w) + U_2(t)v^2(w) + \dots.$$

In light of (1)–(3), and from (17) and (18), we have

$$\begin{aligned} &1 + (1 + \lambda)\Psi_2(\alpha)a_2z + [2(1 + 2\lambda)\Psi_3(\alpha)a_3 - (1 + 3\lambda)\Psi_2^2(\alpha)a_2^2]z^2 + \dots \\ &= 1 + U_1(t)c_1z + [U_1(t)c_2 + U_2(t)c_1^2]z^2 + \dots, \end{aligned}$$

and

$$\begin{aligned} &1 - (1 + \lambda)\Psi_2(\alpha)a_2w + \left\{ [(8\lambda + 4)\Psi_3(\alpha) - (3\lambda + 1)\Psi_2^2(\alpha)]a_2^2 \right. \\ &\quad \left. - 2(1 + 2\lambda)\Psi_3(\alpha)a_3 \right\} w^2 + \dots \\ &= 1 + U_1(t)d_1w + [U_1(t)d_2 + U_2(t)d_1^2]w^2 + \dots. \end{aligned}$$

which yields the following relations:

$$(19) \quad (1 + \lambda)\Psi_2(\alpha)a_2 = U_1(t)c_1,$$

$$(20) \quad -(1 + 3\lambda)\Psi_2^2(\alpha)a_2^2 + 2(1 + 2\lambda)\Psi_3(\alpha)a_3 = U_1(t)c_2 + U_2(t)c_1^2$$

and

$$(21) \quad -(1 + \lambda)\Psi_2(\alpha)a_2 = U_1(t)d_1,$$

$$(22) \quad \begin{aligned} &(4(1 + 2\lambda)\Psi_3(\alpha) - (1 + 3\lambda)\Psi_2^2(\alpha))a_2^2 - \\ &- 2(1 + 2\lambda)\Psi_3(\alpha)a_3 = U_1(t)d_2 + U_2(t)d_1^2. \end{aligned}$$

From (19) and (21) it follows that

$$(23) \quad c_1 = -d_1$$

and

$$(24) \quad 2(1 + \lambda)^2 \Psi_2^2(\alpha) a_2^2 = U_1^2(t)(c_1^2 + d_1^2).$$

Adding (20) and (22), using (24), we obtain

$$a_2^2 = \frac{U_1^3(t)(c_2 + d_2)}{2[\{2(1 + 2\lambda)\Psi_3(\alpha) - (1 + 3\lambda)\Psi_2^2(\alpha)\}U_1^2(t) - (1 + \lambda)^2\Psi_2^2(\alpha)U_2(t)]}.$$

Applying (16) to the coefficients c_2 and d_2 , and using (4) we have

$$(25) \quad |a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{[2(1 + 2\lambda)\Psi_3(\alpha) - (\lambda^2 + 5\lambda + 2)\Psi_2^2(\alpha)]4t^2 + (1 + \lambda)^2\Psi_2^2(\alpha)}}.$$

By subtracting (22) from (20) and using (23),(24), we get

$$a_3 = \frac{U_1^2(t)(c_1^2 + d_1^2)}{2(1 + \lambda)^2\Psi_2^2(\alpha)} + \frac{U_1(c_2 - d_2)}{4(1 + 2\lambda)\Psi_3(\alpha)}.$$

Using (4), once again applying (16) to the coefficients c_1, c_2, d_1 and d_2 , we get

$$(26) \quad |a_3| \leq \frac{4t^2}{(1 + \lambda)^2\Psi_2^2(\alpha)} + \frac{t}{(1 + 2\lambda)\Psi_3(\alpha)}. \quad \square$$

By taking $\lambda = 0$ or $\lambda = 1$ and $t \in (0, 1)$, one can easily state the estimates $|a_2|$ and $|a_3|$ for the function classes $\mathcal{R}_\Sigma(0, \alpha, \Phi) = \mathcal{PS}_\Sigma^*(\alpha, \Phi)$ and $\mathcal{R}_\Sigma(1, \alpha, \Phi) = \mathcal{KS}_\Sigma^*(\alpha, \Phi)$ respectively.

Remark 3. Let f given by (1) be in the class $\mathcal{PS}_\Sigma^*(\alpha, \Phi)$. Then

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{[|\Psi_3(\alpha) - \Psi_2^2(\alpha)|]8t^2 + \Psi_2^2(\alpha)}}$$

and

$$|a_3| \leq \frac{4t^2}{\Psi_2^2(\alpha)} + \frac{t}{\Psi_3(\alpha)}.$$

where $t \neq \frac{\Psi_2(\alpha)}{2\sqrt{2\Psi_2^2(\alpha) - 2\Psi_3(\alpha)}}$.

Remark 4. Let f given by (1) be in the class $\mathcal{KS}_\Sigma^*(\alpha, \Phi)$. Then

$$(27) \quad |a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{[|3\Psi_3(\alpha) - 4\Psi_2^2(\alpha)|]8t^2 + 4\Psi_2^2(\alpha)}}$$

and

$$(28) \quad |a_3| \leq \frac{t^2}{\Psi_2^2(\alpha)} + \frac{t}{3\Psi_3(\alpha)},$$

where $t \neq \frac{\Psi_2(\alpha)}{\sqrt{8\Psi_2^2(\alpha) - 6\Psi_3(\alpha)}}$.

For $\alpha = 0$ Theorem 1 yields the following corollary.

Corollary 1. *Let f given by (1) be in the class $\mathcal{R}_\Sigma(\lambda, 0, \Phi) \equiv \mathcal{K}_\Sigma(\lambda, \Phi)$. Then*

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|(1 + \lambda)^2 - 2(2\lambda^2 + 4\lambda + 1)t^2|}}$$

and

$$|a_3| \leq \frac{t^2}{(1 + \lambda)^2} + \frac{t}{3(1 + 2\lambda)},$$

where $0 \leq \lambda \leq 1$ and $t \neq \frac{1+\lambda}{\sqrt{2(2\lambda^2+4\lambda+1)}}$.

By taking $\alpha = 0$ in the above remarks we get the estimates $|a_2|$ and $|a_3|$ for the function classes $\mathcal{S}_\Sigma^*(\frac{1}{2}, \Phi)$ and $\mathcal{K}_\Sigma^*(\frac{1}{2}, \Phi)$.

Remark 5. Let f given by (1) be in the class $\mathcal{S}_\Sigma^*(\frac{1}{2}, \Phi)$. Then

$$|a_2| \leq 2t\sqrt{2t}$$

and

$$|a_3| \leq 4t^2 + t.$$

Remark 6. Let f given by (1) be in the class $\mathcal{K}_\Sigma^*(\frac{1}{2}, \Phi)$. Then for $t \neq \frac{1}{\sqrt{2}}$,

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|4 - 8t^2|}}$$

and

$$|a_3| \leq t^2 + \frac{t}{3}.$$

3. FEKETE-SZEGÖ INEQUALITY FOR THE FUNCTION CLASS $\mathcal{R}_\Sigma(\lambda, \alpha, \Phi)$

Due to Zaprawa [35], in this section we obtain the Fekete-Szegö inequality for the function classes $\mathcal{R}_\Sigma(\lambda, \alpha, \Phi)$.

Theorem 2. *Let f given by (1) be in the class $\mathcal{R}_\Sigma(\lambda, \alpha, \Phi)$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{t}{(1 + 2\lambda)\Psi_3(\alpha)}, \\ |\mu - 1| \leq \frac{\left| \frac{(1+\lambda)^2\Psi_2^2(\alpha)}{4t^2} + 2(1 + 2\lambda)\Psi_3(\alpha) - (\lambda^2 + 5\lambda + 2)\Psi_2^2(\alpha) \right|}{2(1 + 2\lambda)\Psi_3(\alpha)}, \\ 8|1 - \mu|t^3 \\ \frac{8|1 - \mu|t^3}{|(2(1 + 2\lambda)\Psi_3(\alpha) - (\lambda^2 + 5\lambda + 2)\Psi_2^2(\alpha))4t^2 + (1 + \lambda)^2\Psi_2^2(\alpha)|}, \\ |\mu - 1| \geq \frac{\left| \frac{(1+\lambda)^2\Psi_2^2(\alpha)}{4t^2} + 2(1 + 2\lambda)\Psi_3(\alpha) - (\lambda^2 + 5\lambda + 2)\Psi_2^2(\alpha) \right|}{2(1 + 2\lambda)\Psi_3(\alpha)}. \end{cases}$$

Proof. From (20) and (22)

$$a_3 - \mu a_2^2 = \frac{U_1(t)(c_2 - d_2)}{4(1 + 2\lambda)\Psi_3(\alpha)} + (1 - \mu) \times \frac{U_1^3(t)(c_2 + d_2)}{(4(1 + 2\lambda)\Psi_3(\alpha) - 2(1 + 3\lambda)\Psi_2^2(\alpha))U_1^2(t) - 2U_2(t)(1 + \lambda)^2\Psi_2^2(\alpha)}$$

$$= U_1(t) \left[\left(h(\mu) + \frac{1}{4(1 + 2\lambda)\Psi_3(\alpha)} \right) c_2 + \left(h(\mu) - \frac{1}{4(1 + 2\lambda)\Psi_3(\alpha)} \right) d_2 \right],$$

where

$$h(\mu) = \frac{(1 - \mu)U_1^2(t)}{2[2(1 + 2\lambda)\Psi_3(\alpha) - (1 + 3\lambda)\Psi_2^2(\alpha)]U_1^2(t) - (1 + \lambda)^2\Psi_2^2(\alpha)U_2(t)}.$$

Then, in view of (4), we conclude that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{t}{(1+2\lambda)\Psi_3(\alpha)}, & 0 \leq |h(\mu)| \leq \frac{1}{4(1+2\lambda)\Psi_3(\alpha)}, \\ 4t|h(\mu)|, & |h(\mu)| \geq \frac{1}{4(1+2\lambda)\Psi_3(\alpha)}. \end{cases} \quad \square$$

Taking $\mu = 1$, we have the following corollary.

Corollary 2. *If $f \in \mathcal{R}_\Sigma(\lambda, \alpha, \Phi)$, then*

$$|a_3 - a_2^2| \leq \frac{t}{(1 + 2\lambda)\Psi_3(\alpha)}.$$

Corollary 3. *Let f given by (1) be in the class $\mathcal{S}_\Sigma^*(\alpha, \Phi)$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{t}{\Psi_3(\alpha)}, & |\mu - 1| \leq \frac{\left| \frac{\Psi_2^2(\alpha)}{8t^2} + \Psi_3(\alpha) - \Psi_2^2(\alpha) \right|}{\Psi_3(\alpha)}, \\ \frac{8|1-\mu|t^3}{|((\Psi_3(\alpha) - \Psi_2^2(\alpha))8t^2 + \Psi_2^2(\alpha))|}, & |\mu - 1| \geq \frac{\left| \frac{\Psi_2^2(\alpha)}{8t^2} + \Psi_3(\alpha) - \Psi_2^2(\alpha) \right|}{\Psi_3(\alpha)}. \end{cases}$$

Especially, for $\mu = 1$, if $f \in \mathcal{S}_\Sigma^(\frac{1}{2}, \Phi(z, t))$, we obtain*

$$|a_3 - a_2^2| \leq t.$$

Corollary 4. *Let f given by (1) be in the class $\mathcal{K}_\Sigma^*(\alpha, \Phi)$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{t}{3\Psi_3(\alpha)}, & |\mu - 1| \leq \frac{\left| \frac{\Psi_2^2(\alpha)}{2t^2} + 3\Psi_3(\alpha) - 4\Psi_2^2(\alpha) \right|}{3\Psi_3(\alpha)}, \\ \frac{2|1-\mu|t^3}{|((3\Psi_3(\alpha) - 4\Psi_2^2(\alpha))2t^2 + \Psi_2^2(\alpha))|}, & |\mu - 1| \geq \frac{\left| \frac{\Psi_2^2(\alpha)}{2t^2} + 3\Psi_3(\alpha) - 4\Psi_2^2(\alpha) \right|}{3\Psi_3(\alpha)}. \end{cases}$$

Especially, for $\mu = 1$ if $f \in \mathcal{K}_\Sigma^(\frac{1}{2}, \Phi)$ we obtain*

$$|a_3 - a_2^2| \leq \frac{t}{3}.$$

4. CONFLICTS OF INTEREST

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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H. Ö. GÜNEY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
DICLE UNIVERSITY
TR-21280 DIYARBAKIR
TURKEY
E-mail address: ozlemg@dicle.edu.tr

G. MURUGUSUNDARAMOORTHY,
K. VIJAYA,
K. THILAGAVATHI,
SCHOOL OF ADVANCED SCIENCES
VELLORE INSTITUTE OF TECHNOLOGY
DEEMED TO BE UNIVERSITY
VELLORE – 632014,
INDIA
E-mail address: gmsmoorthy@yahoo.com
kvijaya@vit.ac.in
kthilagavathi@vit.ac.in