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New fixed-circle results on fuzzy metric spaces with an application to dynamic market equilibrium

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Abstract. In this study, the fixed point theory on fuzzy metric spaces has been generalized to the fixed-circle theory by making a geometric interpretation. The necessary conditions to exist the fixed circles of a self-mapping have been investigated and the uniqueness of the circle is examined under suitable conditions. We present some illustrative examples of obtained results and also offer an application to confirm the utility of our established result for finding the unique solution of an integral equation appearing in the dynamic market equilibrium aspects of economics.

1. Introduction and Preliminaries

The fuzzy set theory is one of the most valuable theories in solving uncertainty-related problems. Zadeh put forward this theory in 1965 [27]. Basic definitions and theorems of general topology have been generalized to fuzzy topological spaces. After that, one of this space's primary problems is obtaining an appropriate notion of fuzzy metric spaces. Many authors studied this problem. Significantly, the authors in [1, 3, 8, 11] have presented the concept of fuzzy metric space in various ways. George and Veeramani [4] have defined and studied a concept of fuzzy metric space with the help of continuous t-norm. The study of the relationship between fuzzy metric spaces and metric spaces constitutes a natural and interesting question in the theory of fuzzy metric spaces. In [4], it is proved that every metric space can induce a standard fuzzy metric spaces.

Tas and Özgür [16] have recently pioneered a new trend by approaching the fixed point theory from a different perspective. Using the notion of a fixed circle, they and many researchers obtained valuable results on metric spaces and some generalized metric spaces [10, 12–15, 17–19, 22–25]. Also, Gopal et. al. [7] introduced the notion of a fixed circle in fuzzy metric spaces.

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This paper presents some existence theorems for fixed circles on fuzzy metric spaces with continuous t-norms. Then, since the fixed circles obtained in these theorems do not have to be unique, we give theorems for uniqueness using some contraction conditions. An application to dynamic market equilibrium is presented to validate the main result.

Firstly, we recall some essential background on fuzzy metric spaces.

Definition 1 ([21]). A triangular norm (also called a t-norm) is a binary operation on the unit interval $[0,1]$, i.e., a map $* : [0,1]^2 \to [0,1]$ such that for all $\eta, \zeta, \delta, \varpi \in [0,1]$ the following assumptions are satisfied:

(T1) $\eta * 1 = \eta$;
(T2) $\eta * \zeta = \zeta * \eta$;
(T3) $\eta * (\zeta * \delta) = (\eta * \zeta) * \delta$;
(T4) $\eta * \delta \leq \zeta * \varpi$ whenever $\eta \leq \zeta$ and $\delta \leq \varpi$.

A t-norm is continuous if it is continuous in $[0,1]^2$ as mapping.

Example 1 ([20]). The most well-known continuous t-norm examples are as follows:

- The minimum t-norm: $\eta *_M \zeta = \min\{\eta, \zeta\}$;
- The product t-norm: $\eta *_P \zeta = \eta\zeta$;
- The Lukasiewicz t-norm: $\eta *_L \zeta = \max\{\eta + \zeta - 1, 0\}$;

for all $\eta, \zeta \in [0,1]$.

The next inequalities are satisfied:

$$\eta *_L \zeta \leq \eta *_P \zeta \leq \eta *_M \zeta$$

and

$$\eta * \zeta \leq \eta *_M \zeta$$

for each continuous t-norm $*$.

Definition 2 ([4]). A triple $(\chi, \mathbb{M}, *)$ is said to be a fuzzy metric space when the following conditions are satisfied for all $\eta, \zeta, \delta \in \chi$ and $t, s > 0$, where χ be an arbitrary set, $\mathbb{M}$ is defined as fuzzy set on $\chi \times \chi \times (0, \infty)$:

(FM1) $\mathbb{M}(\eta, \zeta, t) > 0$;
(FM2) $\mathbb{M}(\eta, \zeta, t) = 1$ if and only if $\eta = \zeta$;
(FM3) $\mathbb{M}(\eta, \zeta, t) = \mathbb{M}(\zeta, \eta, t)$;
(FM4) $\mathbb{M}(\eta, \delta, t + s) \geq \mathbb{M}(\eta, \zeta, t) * \mathbb{M}(\zeta, \delta, s)$;
(FM5) $\mathbb{M}(\eta, \zeta, \cdot) : (0, \infty) \to [0,1]$ is continuous.

Lemma 1 ([5]). *$\mathbb{M}(\eta, \zeta, \cdot)$ is non-decreasing for all $\eta, \zeta \in \chi$.*

Example 2 ([4]). Let (χ, d) be a metric space. Denote $\eta *_P \zeta = \eta\zeta$ and let $\mathbb{M}_d$ be a fuzzy set on $\chi \times \chi \times (0, \infty)$ defined as follows:

$$\mathbb{M}_d(\eta, \zeta, t) = \frac{t}{t + d(\eta, \zeta)}$$

for all $\eta, \zeta \in \chi$ and $t > 0$. Then, $(\chi, \mathbb{M}_d, *_P)$ is called as the standard fuzzy metric space induced by d. Even if it is taken $\eta *_M \zeta = \min\{\eta, \zeta\}$, $(\chi, \mathbb{M}_d, *_M)$ will be a fuzzy metric space.

Example 3 ([26])**.** Let $\chi = (0, \infty)$ and $\eta *_P \zeta = \eta\zeta$ for all $\eta, \zeta \in [0, 1]$. Two of the well-known fuzzy metric examples on $(0, \infty)$ are defined by $M(\eta, \zeta, t) = \frac{\min\{\eta,\zeta\}}{\max\{\eta,\zeta\}}$ and $M(\eta, \zeta, t) = \frac{\min\{\eta,\zeta\}+t}{\max\{\eta,\zeta\}+t}$ for all $\eta, \zeta \in \chi$ and for all $t > 0$.

Example 4 ([4])**.** Let $\chi = \mathbb{N}$ and take $\eta *_P \zeta = \eta\zeta$. Define

$$\mathbb{M}(\eta, \zeta, t) = \begin{cases} \frac{\eta}{\zeta}, & \text{if } \eta \le \zeta, \\ \frac{\zeta}{\eta}, & \text{if } \zeta \le \eta, \end{cases}$$

for all $\eta, \zeta \in \chi$ and $t > 0$. Then, $(\chi, \mathbb{M}, *_P)$ is a fuzzy metric space and there exists no metric d on χ satisfying $\mathbb{M}_d(\eta, \zeta, t) = \frac{t}{t+d(\eta,\zeta)}$.

If we take $\eta *_M \zeta = \min\{\eta, \zeta\}$ as $t-$norm, $(\chi, \mathbb{M}, *_M)$ is not a fuzzy metric space. Indeed, if we take $\eta = 2, \zeta = 4$ and $\delta = 5$, then we see that do not satisfy the condition $(FM4)$ as follows:

$$\mathbb{M}(\eta, \delta, t + s) = \tfrac{2}{5} < \min\{\tfrac{2}{4}, \tfrac{4}{5}\} = \min\{\mathbb{M}(\eta, \zeta, t), \mathbb{M}(\zeta, \delta, s)\}.$$

Example 5 ([20])**.** Let χ be the real interval $(2, \infty)$ and think the mapping $\mathbb{M}$ on $\chi \times \chi \times (0, \infty)$ defined as follows:

$$\mathbb{M}(\eta, \zeta, t) = \begin{cases} 1, & \text{if } \eta = \zeta, \\ \frac{1}{\eta} + \frac{1}{\zeta}, & \text{if } \eta \ne \zeta, t > 0, \end{cases}$$

$(\chi, \mathbb{M}, *_L)$ is a fuzzy metric space. On the other hand, if we take $\eta = 1000$, $\zeta = 3$ and $\delta = 10000$, then $(\chi, \mathbb{M}, *_P)$ is not a fuzzy metric space.

Lemma 2 ([20])**.** *Let χ be a non-empty set. If $(\chi, \mathbb{M}, *)$ is a fuzzy metric space and $*'$ is a continuous t-norm such that $*' \le *$, then $(\chi, \mathbb{M}, *')$ is a fuzzy metric space.*

Remark 1. Notice that if we take $\eta *_L \zeta = \max\{\eta + \zeta - 1, 0\}$ as t-norm in Example 4, $(\chi, \mathbb{M}, *_L)$ is a fuzzy metric space.

A circle and a fixed circle are defined on a fuzzy metric space as follows.

Definition 3 ([7])**.** Let $(\chi, \mathbb{M}, *)$ be a fuzzy metric space. A circle of center $\eta_0 \in \chi$ and radius $r \in (0, 1)$ is defined as follows

$$C^F(\eta_0, r, t) = \{\eta \in \chi : \mathbb{M}(\eta, \eta_0, t) = 1 - r\}.$$

This equality can rewrite as follows:

$$C^F(\eta_0, r) = \{\eta \in \chi : \mathbb{M}(\eta, \eta_0, t) = 1 - r, \forall t > 0\}.$$

Example 6. If we take the function $\mathbb{M}_d$ defined in Example 2 with the usual metric space and choose $\eta_0 = 4, r = \frac{1}{3}$ and $t = 2$, then we obtain

$$C^F(4, \tfrac{1}{3}, 2) \;\; = \;\; \{\eta \in \chi : \mathbb{M}(\eta, \eta_0, t) = 1 - r\}$$

$$\begin{aligned} &= \{\eta \in \chi : \mathbb{M}(\eta, 4, 2) = 1 - \tfrac{1}{3}\} \\ &= \{\eta \in \chi : \frac{2}{2 + |\eta - 4|} = \tfrac{2}{3}\} \\ &= \{3, 5\}. \end{aligned}$$

Definition 4 ([7])**.** Let $C^F(\eta_0, r, t)$ be a circle in a fuzzy metric space $(\chi, \mathbb{M}, *)$ and $\tau : \chi \to \chi$ be a self-mapping. If $\tau\eta = \eta$ for all $\eta \in C^F(\eta_0, r, t)$, then the circle $C^F(\eta_0, r, t)$ is called as the fixed circle of τ.

Example 7. Let $\chi = (0, \infty)$ and $\eta *_P \zeta = \eta\zeta$ for all $\eta, \zeta \in [0, 1]$ and let $M(\eta, \zeta, t) = \frac{\min\{\eta,\zeta\}}{\max\{\eta,\zeta\}}$. Define the mapping $\tau : \chi \to \chi$ by

$$\tau\eta = \begin{cases} \frac{\eta^2}{2}, & \text{if } \eta \in (0, 2] \\ \eta, & \text{if } \eta \in (2, 4) \\ \frac{2\eta^2}{9}, & \text{if } \eta \in [4, \infty). \end{cases}$$

It is to verify that the mapping τ fixes the circle $C^F(3, \frac{1}{3}, t)$. However τ does not fixes the circle $C^F(3, \frac{1}{4}, t)$, although the point $\eta = \frac{9}{4}$ is a fixed point of the mapping τ.

The purpose of this work is to present novel fixed-circle results on fuzzy metric spaces taking into account the research and literature mentioned above. Because a fuzzy metric space is a generalization of a metric and there are certain examples of fuzzy metric that are not metric, this is significant (see the Example 4).

2. MAIN RESULTS

2.1. **Some existence conditions for fixed-circles on fuzzy metric spaces.** In order to obtain following existence theorems for a fixed-circle, we used some contractive conditions.

Theorem 1. *Let $(\chi, \mathbb{M}, *_M)$ be a fuzzy metric space where $*_M$ is a minimum t-norm and $C^F(\eta_0, r, t)$ be any circle on χ. Define the mapping*

$$\varphi : \chi \to [0, 1), \varphi_t(\eta) = \mathbb{M}(\eta, \eta_0, t) \tag{1}$$

for all $\eta \in \chi$. If the self-mapping $\tau : \chi \to \chi$ is a function such that for all $\eta \in C^F(\eta_0, r, t)$, the following inequalities are satisfied:

$$\mathbb{M}(\eta, \tau\eta, t) \geq [\varphi_t(\eta) *_M \varphi_t(\tau\eta)] + r \tag{2}$$

$$\mathbb{M}(\tau\eta, \eta_0, t) \geq 1 - r \tag{3}$$

then, the circle $C^F(\eta_0, r, t)$ is a fixed-circle of τ.

Proof. Let $\eta \in C^F(\eta_0, r, t)$. Assume that $\eta \neq \tau\eta$. Then, take into account the inequalities (2) and (3), we obtain that

$$\mathbb{M}(\eta, \tau\eta, t) \geq [\varphi_t(\eta) *_M \varphi_t(\tau\eta)] + r$$

$$
\begin{aligned}
&= \min\{\varphi_t(\eta), \varphi_t(\tau\eta)\} + r \\
&= \min\{\mathbb{M}(\eta, \eta_0, t), \mathbb{M}(\tau\eta, \eta_0, t)\} + r \\
&= \min\{1 - r, \mathbb{M}(\tau\eta, \eta_0, t)\} + r \\
&= 1 - r + r \\
&= 1.
\end{aligned}
$$

Notice that the fuzzy set is defined on $[0, 1]$. Hence, it should be $\mathbb{M}(\eta, \tau\eta, t) = 1$. As a result, we get $\eta = \tau\eta$ and $C^F(\eta_0, r, t)$ is a fixed-circle of τ. □

Example 8. Let $\chi = \mathbb{R}$ endowed with the standard fuzzy metric space with the usual metric space. Think the circle $C^F(0, \frac{2}{5}, 3)$ and define $\tau : \mathbb{R} \to \mathbb{R}$ by

$$\tau\eta = \frac{5\eta + 4\sqrt{7}}{\sqrt{7}\eta + 5}$$

for each $\eta \in \mathbb{R}$. Then, τ verifies the hypotheses of Theorem 1. So, τ fixes the circle $C^F(0, \frac{2}{5}, 3)$.

Theorem 2. *Let $(\chi, \mathbb{M}, *_L)$ be a fuzzy metric space where $*_L$ is a Lukasiewicz t-norm and $C^F(\eta_0, r, t)$ be any circle on χ. Let the mapping φ be defined as Theorem 1. If the self-mapping $\tau : \chi \to \chi$ is a function such that for all $\eta \in C^F(\eta_0, r, t)$ and some $h \in [0, 1)$ the following inequalities are fulfilled:*

$$\mathbb{M}(\eta, \tau\eta, t) \leq \varphi_t(\eta) *_L \varphi_t(\tau\eta) \tag{4}$$

$$\mathbb{M}(\tau\eta, \eta_0, t) - h\mathbb{M}(\eta, \tau\eta, t) \leq r \tag{5}$$

$$\mathbb{M}(\tau\eta, \eta_0, t) \geq r \tag{6}$$

then, the circle $C^F(\eta_0, r, t)$ is a fixed circle of τ.

Proof. By the mapping φ defined in (1), it suffices to show that $\tau\eta = \eta$ where $\eta \in C^F(\eta_0, r, t)$ is an arbitrary point. Taking into consideration the inequalities (4), (5) and (6), we find

$$
\begin{aligned}
\mathbb{M}(\eta, \tau\eta, t) &\leq \varphi_t(\eta) *_L \varphi_t(\tau\eta) \\
&= \max\{\mathbb{M}(\eta, \eta_0, t) + \mathbb{M}(\tau\eta, \eta_0, t) - 1, 0\} \\
&= \mathbb{M}(\eta, \eta_0, t) + \mathbb{M}(\tau\eta, \eta_0, t) - 1 \\
&= 1 - r + \mathbb{M}(\tau\eta, \eta_0, t) - 1 \\
&= \mathbb{M}(\tau\eta, \eta_0, t) - r \\
&\leq h\mathbb{M}(\eta, \tau\eta, t).
\end{aligned}
$$

This gives a contradiction with $h \in [0, 1)$ and shows that $\eta = \tau\eta$. Thus, $C^F(\eta_0, r, t)$ is a fixed-circle of τ. □

Theorem 3. *Let $(\chi, \mathbb{M}, *_L)$ be a fuzzy metric space where $*_L$ is a Lukasiewicz t-norm and $C^F(\eta_0, r, t)$ be any circle on χ. Let the mapping φ be defined as*

Theorem 1. If the self-mapping $\tau : \chi \to \chi$ is a function such that for all $\eta \in C^F(\eta_0, r, t)$, the following inequalities are satisfied:

$$\mathbb{M}(\eta, \tau\eta, t) \leq [\varphi_t(\eta) *_L \varphi_t(\tau\eta)] + 2r \tag{7}$$
$$\mathbb{M}(\tau\eta, \eta_0, t) \leq 1 - r \tag{8}$$

then, the circle $C^F(\eta_0, r, t)$ is a fixed circle of τ.

Proof. By the mapping φ defined in (1), we show that the assumption $\tau\eta \neq \eta$ for all $\eta \in C^F(\eta_0, r, t)$ leads to a contradiction.Taking into consideration the inequalities (7) and (8), we obtain

$$\begin{aligned}
\mathbb{M}(\eta, \tau\eta, t) &\geq [\varphi_t(\eta) *_L \varphi_t(\tau\eta)] + 2 \\
&= \max\{\mathbb{M}(\eta, \eta_0, t) + \mathbb{M}(\tau\eta, \eta_0, t) - 1, 0\} + 2 \\
&= \mathbb{M}(\eta, \eta_0, t) + \mathbb{M}(\tau\eta, \eta_0, t) - 1 + 2 \\
&= 1 - r + \mathbb{M}(\tau\eta, \eta_0, t) + 1 \\
&= 2 + \mathbb{M}(\tau\eta, \eta_0, t) - r \\
&\geq 2 - \mathbb{M}(\eta, \tau\eta, t) \\
2\mathbb{M}(\eta, \tau\eta, t) &\geq 2 \\
\mathbb{M}(\eta, \tau\eta, t) &\geq 1.
\end{aligned}$$

Notice that the fuzzy set is defined $[0, 1]$. Hence, it should be $\mathbb{M}(\eta, \tau\eta, t) = 1$. Consequently, we have $\eta = \tau\eta$ and $C^F(\eta_0, r, t)$ is a fixed-circle of τ. □

Example 9. Let $\chi = \mathbb{N}$ and take $\eta *_L \zeta = \max\{\eta + \zeta - 1, 0\}$. Let think the function $\mathbb{M}$ defined in Example 4. So, $(\mathbb{N}, \mathbb{M}, *_L)$ is a fuzzy metric space. Let $C^F(3, \frac{1}{3}, t)$ be any circle on χ for all $t > 0$. Define $\tau : \mathbb{N} \to \mathbb{N}$ as follows:

$$\tau\eta = \begin{cases} \frac{\eta^2}{2}, & \text{if } \eta \in (0, 3] \\ \eta, & \text{otherwise} \end{cases}$$

for each $\eta \in \mathbb{N}$. Then, the self-mapping τ verifies the hypotheses of Theorem 3. Hence, τ fixes the circle $C^F(3, \frac{1}{3}, t)$.

2.2. **Some uniqueness conditions for fixed circles on fuzzy metric spaces.**

Proposition 1. *If $(\chi, \mathbb{M}, *)$ is a fuzzy metric space with arbitrary circles $C_1^F(\eta_0, r, t)$ and $C_2^F(\eta_1, p, s)$, then there exists at least one self-mapping τ of χ which satisfies τ fixes the circles $C_1^F(\eta_0, r, t)$ and $C_2^F(\eta_1, p, s)$.*

Proof. For any circles $C_1^F(\eta_0, r, t)$ and $C_2^F(\eta_1, p, s)$ on χ, define $\tau : \chi \to \chi$ by

$$\tau\eta = \begin{cases} \eta, & \eta \in C_1^F(\eta_0, r, t) \cup C_2^F(\eta_1, p, s) \\ \alpha, & \text{otherwise} \end{cases}$$

for each $\eta \in \chi$ where α is a constant such that $\mathbb{M}(\alpha, \eta_0, t) \neq 1 - r$ and $\mathbb{M}(\alpha, \eta_1, s) \neq 1 - p$. Define the mapping $\varphi_t, \varphi_s : \chi \to [0, 1]$ as

$$\begin{aligned} \varphi_t(\eta) &= \mathbb{M}(\eta, \eta_0, t), \\ \varphi_s(\eta) &= \mathbb{M}(\eta, \eta_1, s), \end{aligned}$$

for each $\eta \in \chi$ and $t, s > 0$. The hypotheses of Theorem 1 are fulfilled by τ for the circles $C_1^F(\eta_0, r, t)$ and $C_2^F(\eta_1, p, s)$ with the mappings $\varphi_t(\eta)$ and $\varphi_s(\eta)$, respectively. Obviously, $C_1^F(\eta_0, r, t)$ and $C_2^F(\eta_1, p, s)$ are the fixed circles of τ by Theorem 1. □

Example 10. Let $\chi = \mathbb{R}$ endowed with the standard fuzzy metric space with the usual metric space. Consider the circle $C^F(0, \frac{1}{2}, 1)$ on χ and define $\tau : \mathbb{R} \to \mathbb{R}$ as

$$\tau\eta = \begin{cases} \frac{1}{\eta}, & \eta \in \{-1, 1\}, \\ 3, & \text{otherwise}, \end{cases}$$

for each $\eta \in \mathbb{R}$. Then, τ holds the hypotheses of Theorem 1. Hence, τ fixes the circle $C^F(0, \frac{1}{2}, 1)$. But, the circle $C^F(0, \frac{1}{2}, 1)$ is not unique. $C^F(1, \frac{2}{5}, 3)$ is also fixed-circle of τ. One can easily check that τ fulfills the hypotheses of Theorem 1 for the circle $C^F(1, \frac{2}{5}, 3)$.

Therefore, it is crucial to study the uniqueness of the fixed-circles. Next, we provide the uniqueness conditions for the fixed-circles in Theorem 1.

Theorem 4. *Let $(\chi, \mathbb{M}, *)$ be a fuzzy metric space with any circle $C^F(\eta_0, r, t)$ on χ. Under the hypotheses of Theorem 1, also assume the following contractive condition*

$$\mathbb{M}(\tau\eta, \tau\zeta, kt) \geq \mathbb{M}(\eta, \zeta, t) \tag{9}$$

is satisfied for all $\eta \in C^F(\eta_0, r, t), \zeta \in \chi \backslash C^F(\eta_0, r, t)$ and $t > 0, k \in (0, 1)$, then $C^F(\eta_0, r, t)$ is the unique fixed-circle of τ.

Proof. Let $C^F(\eta_0, r, t)$ and $C^F(\eta_1, p, s)$ be two fixed-circles of τ. Let $u \in C^F(\eta_0, r, t)$ and $v \in C^F(\eta_1, p, s)$ be arbitrary points. It suffices to show that $u = v$. Using the inequality (9) and Lemma 1, we obtain

$$\mathbb{M}(u, v, kt) = \mathbb{M}(\tau u, \tau v, kt) \geq \mathbb{M}(u, v, t).$$

This gives a contradiction since $k \in (0, 1)$. As a result, $C^F(\eta_0, r, t)$ is the unique fixed-circle of τ. □

Notice that the contractive inequality (9) τ given in the Proof of Proposition 1 is not satisfied.

Gregori and Sapena [6] proposed a contraction in a fuzzy metric space. Using the contractive condition, we obtain a uniqueness theorem as follows.

Theorem 5. *Let $(\chi, \mathbb{M}, *)$ be a fuzzy metric space with any circle $C^F(\eta_0, r, t)$ on χ. Under the hypotheses of Theorem 1, also assume the following contractive condition*

$$\frac{1}{\mathbb{M}(\tau\eta, \tau\zeta, t)} - 1 \leq k\Big(\frac{1}{\mathbb{M}(\eta, \zeta, t)} - 1\Big)$$

is satisfied for all $\eta \in C^F(\eta_0, r, t), \zeta \in \chi \backslash C^F(\eta_0, r, t), t > 0$, where $k \in (0, 1)$, then $C^F(\eta_0, r, t)$ is the unique fixed-circle of τ.

Proof. The proof is similar to Proof of Theorem 4. □

Remark 2. The choice of used contractive condition in the uniqueness theorem is not unique. Any contractive condition used to derive the fixed point theorem can also be selected.

3. APPLICATION

This section presents an application to dynamic market equilibrium to support our work. In many markets, current prices and price trends affect supply and demand (i.e., whether prices are rising or falling at an increasing or decreasing rate). So, the economist wants to know the current price $P(t)$, the first derivative $\frac{dP(t)}{dt}$ and second derivative $\frac{d^2(P(t))}{dt^2}$. Suppose

$$\mathbb{Q}_s = a_1 + b_1 P(t) + x_1 \frac{dP(t)}{dt} + y_1 \frac{d^2(P(t))}{dt^2} = a_1 + b_1 P + x_1 P' + y_1 P'',$$

$$\mathbb{Q}_d = a_2 + b_2 P(t) + x_2 \frac{dP(t)}{dt} + y_2 \frac{d^2(P(t))}{dt^2} = a_2 + b_2 P + x_2 P' + y_2 P''.$$

a_1, a_2, b_1, b_2, x_1 and x_2 are constants. Comment on the dynamic stability of the market if price clears the market at each point in time. In equilibrium, $\mathbb{Q}_s = \mathbb{Q}_d$ [2]. For this reason,

$$a_1 + b_1 P + x_1 P' + y_1 P'' = a_2 + b_2 P + x_2 P' + y_2 P'',$$

$$(b_1 - b_2)P + (x_1 - x_2)P' + (y_1 - y_2)P'' = -(a_1 - a_2).$$

Letting $b = b_1 - b_2, x = x_1 - x_2, y = y_1 - y_2, a = a_1 - a_2$ and dividing through by y, $P(t)$ is expressed by the following initial value problem

$$\begin{cases} P'' + \frac{x}{y}P' + \frac{b}{y}P = -\frac{a}{y} \\ P(0) = 0 \\ P'(0) = 0 \end{cases} \tag{10}$$

where $\frac{x^2}{y} = \frac{4b}{y}$ and $\frac{b}{x} = \mu$ is a continuous function.

One can easily show that problem (10) is equivalent to the integral equation:

$$P(t) = \int_0^I \Gamma(t, s) F(t, s, P(s)) ds,$$

where $\Gamma(t,s)$ is Green's function given by

$$\Gamma(t,s)=\begin{cases} se^{\frac{\mu}{2}}(t-s) & \text{if } 0\leq s\leq t\leq I \\ te^{\frac{\mu}{2}}(s-t) & \text{if } 0\leq t\leq s\leq I. \end{cases}$$

To prove the existence part of a solution integral equation, we use Theorem 4:

$$P(t)=\int_0^I G(t,s,P(s))ds. \tag{11}$$

Let $\chi=C([0,I])$ be the set of real continuous functions defined on the interval $[0,I]$. Define

$$M(\eta,\zeta,t)=\sup_{t\in[0,I]}\frac{\min\{\eta,\zeta\}+t}{\max\{\eta,\zeta\}+t}$$

for each $t>0$, and $\eta,\zeta\in\chi$ with continuous t-norm $*$ such that $\eta *_P \zeta=\eta\zeta$. One can easily verify that $(\chi,M,*)$ is a fuzzy metric space. Consider the mapping $\tau:\chi\to\chi$ is defined by

$$\tau P(t)=\int_0^I G(t,s,P(s))ds.$$

Theorem 6. *Consider the equation* (11) *and suppose that the following statements hold:*

(1) $G:[0,I]\times[0,I]\to\mathbb{R}^+$ *is a continuous function,*
(2) *There exists a continuous function* $\Gamma:[0,I]\times[0,I]\to\mathbb{R}^+$ *such that*

$$\sup_{t\in[0,I]}\int_0^I \Gamma(t,s)ds\geq 1,$$

(3)
$$\int_0^I \min\{G(t,s,\eta(s)),G(t,s,\zeta(s))\}ds+kt\geq \int_0^I \Gamma(t,s)\min\{\eta(s),\zeta(s)\}ds+t,$$
$$\int_0^I \max\{G(t,s,\eta(s)),G(t,s,\zeta(s))\}ds+kt\geq \int_0^I \Gamma(t,s)\max\{\eta(s),\zeta(s)\}ds+t.$$

Then, the integral equation (11) *has a unique solution.*

Proof. For $\eta,\zeta\in\chi$, by using of assumptions of Theorem 6, we have

$$\begin{aligned} M(\tau\eta,\tau\zeta,kt) &= \sup_{t\in[0,I]}\frac{\min\{\int_0^I G(t,s,\eta(s))ds,\int_0^I G(t,s,\zeta(s))ds\}+kt}{\max\{\int_0^I G(t,s,\eta(s))ds,\int_0^I G(t,s,\zeta(s))ds\}+kt} \\ &= \sup_{t\in[0,I]}\frac{\int_0^I \min\{G(t,s,\eta(s)),G(t,s,\zeta(s))\}ds+kt}{\int_0^I \max\{G(t,s,\eta(s)),G(t,s,\zeta(s))\}ds+kt} \end{aligned}$$

$$
\begin{aligned}
&\geq \sup_{t\in[0,I]} \frac{\int_0^I \Gamma(t,s)\min\{\eta(s),\zeta(s)\}ds + t}{\int_0^I \Gamma(t,s)\max\{\eta(s),\zeta(s)\}ds + t} \\
&\geq \sup_{t\in[0,I]} \frac{\min\{\eta(s),\zeta(s)\}\int_0^I \Gamma(t,s)ds + t}{\max\{\eta(s),\zeta(s)\}\int_0^I \Gamma(t,s)ds + t} \\
&\geq \frac{\min\{\eta(s),\zeta(s)\} + t}{\min\{\eta(s),\zeta(s)\} + t} \\
&= M(\eta,\zeta,t).
\end{aligned}
$$

Thus, $M(\tau\eta,\tau\zeta,t) \geq M(\eta,\zeta,t)$ for all $\eta,\zeta \in \chi$ and all conditions of Theorem 4 are satisfied. Consequently, the equation (11) has a unique fixed circle. □

4. Conclusions

In this study, the geometric properties of the fixed point set are investigated when the fixed point of the mappings on fuzzy metric spaces is more than one. For this, the fixed-circle problem, which is a generalization of the fixed point theory, is used. By this new approach, new geometric generalizations of known fixed point theorems can study on fuzzy metric and generalized metric spaces.

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