

Fixed point theorems in complex valued b -metric spaces

DINANATH BARMAN, KRISHNADHAN SARKAR*,
KALISHANKAR TIWARY

ABSTRACT. In this paper, we have proved common fixed point theorems using Hardy and Rogers type contraction condition in complex-valued b -metric spaces. The results of the paper extend the results proved in S. Ali [1].

1. INTRODUCTION

In 1922, Banach first proved a fixed point theorem in a complete metric space. This theorem is known as Banach's fixed point theorem. After the work of Banach, many researchers ([6, 11, 12, 15], etc.) have proved several fixed point theorems in many branches of mathematics. The notion of complex-valued metric space was introduced by Azam et al. [2]. Rao et al. [14] extended the notion of complex-valued metric space to complex-valued b -metric space. Dubey et al. [7], Berrah et al. [3], Dubey and Tripathi [8], Ali [1], Sitthikul and Saejung [19], Singh et al. [18], Rouzkard and Imdad [16], Bhardwaj and Wadhwa [4], Hamaizia and Murthy [9], Saluja [17], Bouhadjera [5] have proved several fixed point theorems in complex valued metric spaces and complex-valued b -metric spaces using different conditions on the operators.

It is further observed that Hardy and Rogers [10] have extended Banach fixed point theorem in complete metric spaces. Hardy and Rogers' notions have also been generalized by various researchers. Recently, Mukheimer [13] has proved a uniqueness common fixed point in complete complex valued b -metric spaces.

In this paper we have proved some common fixed theorems using Hardy and Rogers type contraction mappings. Our theorems have generalized the available results in [1].

2020 *Mathematics Subject Classification.* Primary: 54H25; Secondary: 47H10.

Key words and phrases. Complex valued b -metric space.

Full paper. Received 4 August 2022, accepted 7 March 2023, available online 19 May 2023.

*Corresponding author.

2. PRELIMINARIES

With the usual notation $\mathbb{N}, \mathbb{R}, \mathbb{C}$, let $z_1, z_2 \in \mathbb{C}$, we define a partial order \preceq on \mathbb{C} as follows:

$$z_1 \preceq z_2 \text{ if and only if } \operatorname{Re}z_1 \leq \operatorname{Re}z_2 \text{ and } \operatorname{Im}z_1 \leq \operatorname{Im}z_2.$$

Thus we can say, $z_1 \preceq z_2$ if one of the following holds:

- (i) $\operatorname{Re}z_1 = \operatorname{Re}z_2$ and $\operatorname{Im}z_1 = \operatorname{Im}z_2$,
- (ii) $\operatorname{Re}z_1 = \operatorname{Re}z_2$ and $\operatorname{Im}z_1 < \operatorname{Im}z_2$,
- (iii) $\operatorname{Re}z_1 < \operatorname{Re}z_2$ and $\operatorname{Im}z_1 = \operatorname{Im}z_2$,
- (iv) $\operatorname{Re}z_1 < \operatorname{Re}z_2$ and $\operatorname{Im}z_1 < \operatorname{Im}z_2$.

We write $z_1 \prec z_2$ if $z_1 \neq z_2$ and any one of (ii), (iii) and (iv) is satisfied. If only the condition (iv) hold, then we write $z_1 \prec z_2$.

It is clear that

- (i) $z_1 \preceq z_2$ and $z_2 \prec z_3$ implies $z_1 \prec z_3$,
- (ii) $a, b \in \mathbb{R}$ and $a < b$, then $az \preceq bz$, for all $z \in \mathbb{C}$,
- (iii) $0 \preceq z_1 \preceq z_2$, then $|z_1| < |z_2|$.

3. DEFINITIONS

Azam et al. [2] defined the complex valued metric space as follows.

Definition 1. A complex valued metric on a non-empty set X is a mapping $d : X \times X \rightarrow \mathbb{C}$ such that for all $x, y, z \in X$, the following conditions holds:

- (i) $0 \preceq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,
- (ii) $d(x, y) = d(y, x)$,
- (iii) $d(x, y) \preceq d(x, z) + d(z, y)$.

Then the pair (X, d) is called a complex valued metric space.

Definition 2 ([14]). A complex valued metric on a non-empty set X is a mapping $d : X \times X \rightarrow \mathbb{C}$, such that for all $x, y, z \in X$, the following conditions holds:

- (i) $0 \preceq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,
- (ii) $d(x, y) = d(y, x)$,
- (iii) there exists a real number $s \geq 1$ such that $d(x, y) \preceq s[d(x, z) + d(z, y)]$.

Then the pair (X, d) is called a complex valued b -metric space with coefficient $s \geq 1$.

Example 1 ([14]). Let $X = [0, 1]$. Define the mapping $d : X \times X \rightarrow \mathbb{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$, for all $x, y \in X$. Then (X, d) is a complex valued b -metric space with $s = 2$.

Definition 3 ([14]). Let (X, d) be a complex valued b -metric space and $A \subset X$. We recall the following definitions:

- (i) $a \in A$ is called an interior point of the set A whenever there is $0 < r \in \mathbb{C}$, such that

$$N(a, r) \subset A,$$

where $N(a, r) = \{x \in X : d(a, y) < r\}$.

- (ii) A point $x \in X$ is called a limit point of A whenever for every $0 < r \in \mathbb{C}$,

$$N(x, r) \cap (A \setminus \{x\}) \neq \phi.$$

- (iii) A subset $A \subset X$ is called open whenever each element of A is an interior point of A .
- (iv) A subset $A \subset X$ is called closed whenever each limit point of A belongs to A .

The collection $F = \{N(x, r) : x \in X, 0 < r\}$ is a sub-basis for a topology on X . The topology is denoted by τ . It is to be noted that this topology τ is Hausdorff topology.

Definition 4 ([14]). Let (X, d) be a complex valued b -metric space and $\{x_n\}$ be a sequence in X and $x \in X$. We call

- (i) the sequence $\{x_n\}$ converges to x if for every $c \in \mathbb{C}$ with $0 < c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) < c$. We write this as $\lim_{n \rightarrow \infty} x_n = x$ or, $x_n \rightarrow x$ as $n \rightarrow \infty$;
- (ii) The sequence $\{x_n\}$ is a Cauchy sequence if for every $c \in \mathbb{C}$ with $0 < c$ there is $N \in \mathbb{N}$ such that for all $n > N$ and $m \in \mathbb{N}$, $d(x_n, x_m) < c$;
- (iii) The metric space (X, d) is a complete complex valued b -metric space if every Cauchy sequence is convergent in X .

Azam et al. [2] established the following lemmas.

Lemma 1. *Let (X, d) be a complex valued b -metric space with coefficient $s \geq 1$ and $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \rightarrow 0$ as $n \rightarrow \infty$.*

Lemma 2. *Let (X, d) be a complex valued b -metric space with coefficient $s \geq 1$ and $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \rightarrow 0$ as $n, m \rightarrow \infty$.*

4. MAIN RESULTS

Our main results are as follows.

Theorem 1. *Let (X, d) be a complete complex valued b -metric space with coefficient $s \geq 1$ and $f, g : X \rightarrow X$ be self-maps satisfying the following condition:*

$$(1) \quad d(fx, gy) \preceq \alpha d(x, y) + \beta \max \left\{ d(x, y), \frac{d(x, fx)d(y, gy)}{1 + d(fx, gy)} \right\} \\ + \gamma \min \{ d(x, gy), d(y, fx) \},$$

where $\alpha + \beta + s\gamma < 1$, $\alpha, \beta, \gamma \geq 0$. Then f and g have unique common fixed point in X .

Proof. Let $x_0 \in X$ be an arbitrary. We construct a sequence $\{x_n\}$ in X such that

$$x_{2n+1} = fx_{2n}, \quad x_{2n+2} = gx_{2n+1}.$$

Now,

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(fx_{2n}, gx_{2n+1}) \\ &\preceq \alpha d(x_{2n}, x_{2n+1}) \\ &\quad + \beta \max \left\{ d(x_{2n}, x_{2n+1}), \frac{d(x_{2n}, fx_{2n})d(x_{2n+1}, gx_{2n+1})}{1 + d(fx_{2n}, gx_{2n+1})} \right\} \\ &\quad + \gamma \min \{ d(x_{2n}, gx_{2n+1}), d(x_{2n+1}, fx_{2n}) \} \\ &= \alpha d(x_{2n}, x_{2n+1}) \\ &\quad + \beta \max \left\{ d(x_{2n}, x_{2n+1}), \frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n+1}, x_{2n+2})} \right\} \\ &\quad + \gamma \min \{ d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1}) \} \\ &= \alpha d(x_{2n}, x_{2n+1}) + \beta d(x_{2n}, x_{2n+1}) + \gamma \cdot 0 \\ &= (\alpha + \beta)d(x_{2n}, x_{2n+1}). \end{aligned}$$

Therefore,

$$\begin{aligned} |d(x_{2n+1}, x_{2n+2})| &\leq |(\alpha + \beta)d(x_{2n}, x_{2n+1})| \\ &\leq (\alpha + \beta)^2 |d(x_{2n-1}, x_{2n})| \\ &\leq \dots \\ &\leq (\alpha + \beta)^{2n+1} |d(x_0, x_1)|. \end{aligned}$$

Thus,

$$\lim_{n \rightarrow \infty} |d(x_{2n+1}, x_{2n+2})| = 0 \text{ [since } \alpha + \beta < 1 \text{].}$$

Again let, $n, m \in \mathbb{N}, n \geq m$. Then,

$$\begin{aligned} d(x_{n+1}, x_{m+1}) &= d(fx_n, gx_m) \\ &\preceq \alpha d(x_n, x_m) + \beta \max \left\{ d(x_n, x_m), \frac{d(x_n, fx_n)d(x_m, gx_m)}{1 + d(fx_n, gx_m)} \right\} \\ &\quad + \gamma \min \{ d(x_n, gx_m), d(x_m, fx_n) \} \\ &= \alpha d(x_n, x_m) + \beta \max \left\{ d(x_n, x_m), \frac{d(x_n, x_{n+1})d(x_m, x_{m+1})}{1 + d(x_{n+1}, x_{m+1})} \right\} \\ &\quad + \gamma \min \{ d(x_n, x_{m+1}), d(x_m, x_{n+1}) \} \\ &\preceq \alpha d(x_n, x_m) + \beta \max \left\{ d(x_n, x_m), \frac{d(x_n, x_{n+1})d(x_m, x_{m+1})}{1 + d(x_{n+1}, x_{m+1})} \right\} \\ &\quad + \gamma \min \{ s[d(x_n, x_m) + d(x_m, x_{m+1})], s[d(x_m, x_n) + d(x_n, x_{n+1})] \}. \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} |d(x_{n+1}, x_{m+1})| \leq (\alpha + \beta) \lim_{n \rightarrow \infty} |d(x_n, x_m)| + \lim_{n \rightarrow \infty} \gamma s |d(x_n, x_m)|$$

implies

$$\lim_{n \rightarrow \infty} |d(x_n, x_m)| \leq (\alpha + \beta + s\gamma) \lim_{n \rightarrow \infty} |d(x_n, x_m)|$$

implies

$$\lim_{n \rightarrow \infty} |d(x_n, x_m)| = 0.$$

Thus $\{x_n\}$ is a Cauchy sequence. Since X is a complete complex valued b -metric space, there exists an $u \in X$ such that

$$\lim_{n \rightarrow \infty} x_n = u.$$

Therefore,

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} x_{n-1} = u = \lim_{n \rightarrow \infty} gx_n.$$

Now,

$$\begin{aligned} d(fu, u) &\leq s[d(fu, x_{n+1}) + d(x_{n+1}, u)] \\ &= s d(fu, gx_n) + sd(x_{n+1}, u) \\ &\leq s[\alpha d(u, x_n) + \beta \max\left\{d(u, x_n), \frac{d(u, fu)d(x_n, gx_n)}{1 + d(fu, gx_n)}\right\} \\ &\quad + \gamma \min\{d(u, gx_n), d(x_n, fu)\}] + sd(x_{n+1}, u) \\ &= s[\alpha d(u, x_n) + \beta \max\left\{d(u, x_n), \frac{d(u, fu)d(x_n, x_{n+1})}{1 + d(fu, x_{n+1})}\right\} \\ &\quad + \gamma \min\{d(u, x_{n+1}), d(x_n, fu)\}] + sd(x_{n+1}, u), \end{aligned}$$

which implies, $\lim_{n \rightarrow \infty} |d(fu, u)| \rightarrow 0$.

Thus, $|d(fu, u)| = 0$ implies $fu = u$. So u is a fixed point of f .

Again,

$$\begin{aligned} d(u, gu) &\leq s[d(u, x_{n+1}) + d(x_{n+1}, gu)] \\ &= sd(u, x_{n+1}) + sd(fx_n, gu) \\ &\leq sd(u, x_{n+1}) + s[\alpha d(x_n, u) + \beta \max\left\{d(x_n, u), \frac{d(x_n, fx_n)d(u, gu)}{1 + d(fx_n, gu)}\right\} \\ &\quad + \gamma \min\{d(x_n, gu), d(u, fx_n)\}] \\ &= sd(u, x_{n+1}) + s[\alpha d(x_n, u) + \beta \max\left\{d(x_n, u), \frac{d(x_n, x_{n+1})d(u, gu)}{1 + d(x_{n+1}, gu)}\right\} \\ &\quad + \gamma \min\{d(x_n, gu), d(u, x_{n+1})\}], \end{aligned}$$

which implies

$$\lim_{n \rightarrow \infty} |d(u, gu)| = 0,$$

implies

$$gu = u.$$

Thus u is a common fixed point of f and g .

Let, v be another common fixed point f and g . Then,

$$\begin{aligned} d(u, v) &= d(fu, gv) \\ &\preceq \alpha d(u, v) + \beta \max \left\{ d(u, v), \frac{d(u, fu)d(v, gv)}{1 + d(fu, gv)} \right\} \\ &\quad + \gamma \min \{ d(u, gv), d(v, fu) \} \\ &= \alpha d(u, v) + \beta \max \left\{ d(u, v), \frac{d(u, u)d(v, v)}{1 + d(u, v)} \right\} \\ &\quad + \gamma \min \{ d(u, v), d(v, u) \} \\ &= (\alpha + \beta + \gamma)d(u, v) \end{aligned}$$

implies

$$(1 - \alpha - \beta - \gamma)|d(u, v)| = 0,$$

implies

$$|d(u, v)| = 0,$$

i.e., $u = v$.

Thus f and g have unique common fixed point in X . \square

Corollary 1. Let (X, d) be a complete complex valued b -metric space with coefficient $s \leq 1$ and $f, g : X \rightarrow X$ be self-maps satisfying the following condition:

$$d(fx, gy) \preceq \beta \max \left\{ d(x, y), \frac{d(x, fx)d(y, gy)}{1 + d(fx, gy)} \right\},$$

where $0 \leq \beta < 1$. Then f and g have unique common fixed point in X .

This result is **Theorem 1** of S. Ali [1].

Corollary 2. Let (X, d) be a complete complex valued b -metric space with coefficient $s \geq 1$ and $f : X \rightarrow X$ be self-map satisfying the following condition:

$$\begin{aligned} d(fx, fy) &\preceq \alpha d(x, y) + \beta \max \left\{ d(x, y), \frac{d(x, fx)d(y, fy)}{1 + d(fx, fy)} \right\} \\ &\quad + \gamma \min \{ d(x, fy), d(y, fx) \}, \end{aligned}$$

where $\alpha + \beta + s\gamma < 1$, $\alpha, \beta, \gamma \geq 0$. Then f have unique fixed point in X .

Corollary 3. Let (X, d) be a complete complex valued b -metric space with coefficient $s \geq 1$ and $f : X \rightarrow X$ be self-map satisfying the following condition:

$$d(fx, fy) \preceq \alpha d(x, y) + \beta \max \left\{ d(x, y), \frac{d(x, fx)d(y, fy)}{1 + d(fx, fy)} \right\},$$

where $\alpha + \beta < 1$, $\alpha, \beta \geq 0$.

Corollary 4. Let (X, d) be a complete complex valued b -metric space with coefficient $s \geq 1$ and $f : X \rightarrow X$ be self-map satisfying the following condition:

$$d(fx, fy) \preceq \alpha d(x, y),$$

where $0 \leq \alpha < 1$. Then f have a unique fixed point in X .

This result is **Banach Theorem** in complete complex valued b -metric space.

Example 2. Let $X = \mathbb{C}$ and $d : X \times X \rightarrow \mathbb{C}$ be defined by $d(x, y) = i|x - y|^2$. Also let $f, g : X \rightarrow X$ be given by $fx = \frac{x}{2}, gx = \frac{x}{3}$.

Then clearly

- (i) $0 \preceq i|x - y|^2 = d(x, y)$ and $d(x, y) = i|x - y|^2 = 0$ if and only if $|x - y| = 0$ i.e., $x = y$.
- (ii) $d(x, y) = d(y, x)$.
- (iii) $d(x, y) = i|x - y|^2 = i|(x - z) + (z - y)|^2 \preceq i\{|x - z|^2 + |z - y|^2 + 2|x - z||z - y|\} \preceq 2i[|x - z|^2 + |z - y|^2] = 2[d(x - z) + d(z - y)]$.

Thus (X, d) is a complex valued b -metric space with coefficient $s = 2$.

Now consider the sequence $\{x_n\}$, where $x_n = \frac{1}{n+1}$ for $i = 0, 1, 2, \dots$, with initial approximation $x_0 = 1$ given by $x_n = fx_{n-1}$ and $x_{n+1} = gx_n$.

Again,

$$\begin{aligned} d(x, y) &= i|x - y|^2, \\ d(fx, gy) &= i|fx - gy|^2 = i\left|\frac{x}{2} - \frac{y}{3}\right|^2, \\ d(x, fx) &= i|x - fx|^2 = i\left|x - \frac{x}{2}\right|^2 = i\left|\frac{x}{2}\right|^2, \\ d(y, fy) &= i|y - gy|^2 = i\left|y - \frac{y}{3}\right|^2 = i\left|\frac{2y}{3}\right|^2, \\ d(x, gx) &= i|x - gx|^2 = i\left|x - \frac{x}{3}\right|^2, \\ d(y, fy) &= i|y - fy|^2 = i\left|y - \frac{y}{2}\right|^2. \end{aligned}$$

Since

$$\begin{aligned} d(x, fx)d(y, fy) &= i\left|\frac{x}{2}\right|^2 i\left|\frac{2y}{3}\right|^2 = -\left|\frac{xy}{9}\right|, \\ \max\left\{d(x, y), \frac{d(x, fx)d(y, fy)}{1 + d(fx, gy)}\right\} &= d(x, y) = i|x - y|^2. \end{aligned}$$

Also, $\min\{d(x, gy), d(y, fx)\} \preceq d(x, y)$. Therefore the condition of (1) is satisfied. So by **Theorem 1**, f and g have unique common fixed point $'0 + i0'$.

Theorem 2. Let (X, d) be a complete complex valued b -metric space with coefficient $s \geq 1$ and $f : X \rightarrow X$ be self-map satisfying the following condition:

$d(fx, fy) \preceq \alpha_1 d(x, y) + \alpha_2 d(x, fx) + \alpha_3 d(y, fy) + \alpha_4 d(x, fy) + \alpha_5 d(y, fx)$, where each of $\alpha_i \geq 0$ and $\alpha_1 + s\alpha_2 + \alpha_3 + 2s\alpha_4 + s\alpha_5 < 1$. Then f have a unique fixed point in X .

Proof. Let $x_0 \in X$ be an initial point. We construct a sequence $\{x_n\} \in X$ such that $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$.

At first we show that $\lim_{n \rightarrow \infty} |d(x_n, x_{n+1})| = 0$.

Since,

$$\begin{aligned}
 d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\
 &\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, fx_{n-1}) + \alpha_3 d(x_n, fx_n) \\
 &\quad + \alpha_4 d(x_{n-1}, fx_n) + \alpha_5 d(x_n, fx_{n-1}) \\
 &= \alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, x_n) + \alpha_3 d(x_n, x_{n+1}) \\
 &\quad + \alpha_4 d(x_{n-1}, x_{n+1}) + \alpha_5 d(x_n, x_n) \\
 &\leq (\alpha_1 + \alpha_2) d(x_{n-1}, x_n) + \alpha_3 d(x_n, x_{n+1}) \\
 &\quad + \alpha_4 s [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + \alpha_5 \cdot 0 \\
 &= (\alpha_1 + \alpha_2 + s\alpha_4) d(x_{n-1}, x_n) + (\alpha_3 + s\alpha_4) d(x_n, x_{n+1})
 \end{aligned}$$

which implies

$$(1 - \alpha_3 - s\alpha_4) d(x_n, x_{n+1}) \leq (\alpha_1 + \alpha_2 + s\alpha_4) d(x_{n-1}, x_n)$$

implies

$$\begin{aligned}
 d(x_{n-1}, x_n) &\leq \left(\frac{\alpha_1 + \alpha_2 + s\alpha_4}{1 - \alpha_3 - s\alpha_4} \right) d(x_{n-1}, x_n) \\
 &= kd(x_{n-1}, x_n), \quad \text{where } k = \left(\frac{\alpha_1 + \alpha_2 + s\alpha_4}{1 - \alpha_3 - s\alpha_4} \right) \\
 &\leq k^2 d(x_{n-2}, x_{n-1}) \\
 &\quad \vdots \\
 &\leq k^n d(x_0, x_1).
 \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} |d(x_n, x_{n+1})| = 0.$$

Now let, $n, m \in \mathbb{N}$ and $n \geq m$. Then

$$\begin{aligned}
 d(x_m, x_n) &= d(fx_{m-1}, fx_{n-1}) \\
 &\leq \alpha_1 d(x_{m-1}, x_{n-1}) + \alpha_2 d(x_{m-1}, fx_{m-1}) + \alpha_3 d(x_{n-1}, fx_{n-1}) \\
 &\quad + \alpha_4 d(x_{m-1}, fx_{n-1}) + \alpha_5 d(x_{n-1}, fx_{m-1}) \\
 &= \alpha_1 d(x_{m-1}, x_{n-1}) + \alpha_2 d(x_{m-1}, x_m) + \alpha_3 d(x_{n-1}, x_n) + \\
 &\quad \alpha_4 d(x_{m-1}, x_n) + \alpha_5 d(x_{n-1}, x_m) \\
 &\leq \alpha_1 d(x_{m-1}, x_{n-1}) + \alpha_2 d(x_{m-1}, x_m) + \alpha_3 d(x_{n-1}, x_n) \\
 &\quad + \alpha_4 s [d(x_{m-1}, x_m) + d(x_m, x_n)] \\
 &\quad + \alpha_5 s [d(x_{n-1}, x_n) + d(x_n, x_m)].
 \end{aligned}$$

Taking modulus and limit as $n \rightarrow \infty$, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} |d(x_m, x_n)| &\leq \alpha_1 \lim_{n \rightarrow \infty} |d(x_{m-1}, x_{n-1})| + \alpha_2 \cdot 0 + \alpha_3 \cdot 0 \\ &\quad + (\alpha_4 s + \alpha_5 s) \lim_{n \rightarrow \infty} |d(x_m, x_n)| \end{aligned}$$

implies

$$(1 - \alpha_1 - \alpha_4 s - \alpha_5 s) \lim_{n \rightarrow \infty} |d(x_m, x_n)| \leq 0$$

implies

$$\lim_{n \rightarrow \infty} |d(x_m, x_n)| = 0.$$

Thus $\{x_n\}$ is a Cauchy sequence in X . Since the space is complete, there exists an $x \in X$ such that $\lim_{n \rightarrow \infty} |d(x_n, x)| = 0$. Now we show that x is a fixed point of f .

Again,

$$\begin{aligned} d(fx, x) &\preceq s[d(fx, fx_n) + d(fx_n, x)] \\ &\lesssim s[\alpha_1 d(x, x_n) + \alpha_2 d(x, fx) + \alpha_3 d(x_n, fx_n) + \alpha_4 d(x, fx_n) \\ &\quad + \alpha_5 d(x_n, fx) + d(x_{n+1}, x)] \\ &= s[\alpha_1 d(x, x_n) + \alpha_2 d(x, fx) + \alpha_3 d(x_n, x_{n+1}) + \alpha_4 d(x, x_{n+1}) \\ &\quad + \alpha_5 d(x_n, fx) + d(x_{n+1}, x)] \end{aligned}$$

implies

$$(2) \quad \begin{aligned} \lim_{n \rightarrow \infty} |d(fx, x)| &\leq s[\alpha_1 \cdot 0 + \alpha_2 |d(fx, x)| + \alpha_3 \cdot 0 + \alpha_4 \cdot 0 \\ &\quad + \alpha_5 \lim_{n \rightarrow \infty} |d(x_n, fx)| + 0]. \end{aligned}$$

Again,

$$(3) \quad \begin{aligned} d(x_n, fx) &= d(fx_{n-1}, fx) \\ &\preceq \alpha_1 d(x_{n-1}, x) + \alpha_2 d(x_{n-1}, fx_{n-1}) + \alpha_3 d(x, fx) \\ &\quad + \alpha_4 d(x_{n-1}, fx) + \alpha_5 d(x, fx_{n-1}) \\ &= \alpha_1 d(x_{n-1}, x) + \alpha_2 d(x_{n-1}, x_n) + \alpha_3 d(x, fx) \\ &\quad + \alpha_4 d(x_{n-1}, fx) + \alpha_5 d(x, x_n). \end{aligned}$$

If $d(x_{n-1}, fx) \preceq d(x, fx)$, then from (3) we have

$$d(x_n, fx) \preceq \alpha_1 d(x_{n-1}, x) + \alpha_2 d(x_{n-1}, x_n) + (\alpha_3 + \alpha_4) d(x, fx) + \alpha_5 d(x_n, x).$$

Therefore,

$$\lim_{n \rightarrow \infty} |d(x_n, fx)| \leq (\alpha_3 + \alpha_4) |d(x, fx)|.$$

From (2), we get

$$\lim_{n \rightarrow \infty} |d(x, fx)| \leq (\alpha_2 + \alpha_3 + \alpha_4) |d(x, fx)|$$

implies $|d(x, fx)| = 0$, i.e., $fx = x$.

Again if $d(x, fx) \preceq d(x_{n-1}, fx)$, then from (3), we get
 $d(x_n, fx) \preceq \alpha_1 d(x_{n-1}, x) + \alpha_2 d(x_{n-1}, x_n) + (\alpha_3 + \alpha_4) d(x_{n-1}, fx) + \alpha_5 d(x_n, x)$.
 Therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} |d(x_n, fx)| &\leq (\alpha_3 + \alpha_4) \lim_{n \rightarrow \infty} d(x_{n-1}, fx) \\ &\leq (\alpha_3 + \alpha_4)^2 \lim_{n \rightarrow \infty} d(x_{n-2}, fx) \\ &\quad \vdots \\ &\leq (\alpha_3 + \alpha_4)^{n-1} \lim_{n \rightarrow \infty} d(x_0, fx). \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} |d(x_n, fx)| = 0.$$

Thus we get from (2),

$$|d(fx, x)| \leq s\alpha_2 |d(x, fx)|$$

implies

$$(1 - \alpha_2 s) |d(fx, x)| \leq 0$$

implies $|d(fx, x)| = 0$, i.e., $fx = x$. Therefore, F have a fixed point.

To show that x is unique let, y be another fixed point of f . Then we get

$$\begin{aligned} d(x, y) &= d(fx, fy) \\ &\preceq \alpha_1 d(x, y) + \alpha_2 d(x, fx) + \alpha_3 d(y, fy) + \alpha_4 d(x, fy) + \alpha_5 d(y, fx) \\ &= \alpha_1 d(x, y) + \alpha_2 d(x, x) + \alpha_3 d(y, y) + \alpha_4 d(x, y) + \alpha_5 d(y, x) \end{aligned}$$

implies

$$(1 - \alpha_1 - \alpha_4 - \alpha_5) |d(x, y)| = 0$$

implies, $x = y$.

Thus f have a unique fixed point in X . □

Corollary 5. *Let (X, d) be a complete complex valued b -metric space with coefficient $s \geq 1$ and $f : X \rightarrow X$ be self-map satisfying the following condition:*

$$d(fx, fy) \preceq \alpha_1 d(x, y),$$

where each of $0 \leq \alpha_1 < 1 \leq 0$. Then f have a unique fixed point in X .

This result is **Banach** contraction condition in complex valued b -metric space.

Corollary 6. *Let (X, d) be a complete complex valued b -metric space with coefficient $s \geq 1$ and $f : X \rightarrow X$ be self-map satisfying the following condition:*

$$d(fx, fy) \preceq \alpha_2 [d(x, fx) + d(y, fy)],$$

where each of $\alpha_2 \geq 0$ and $s\alpha_2 = \alpha_3 < \frac{1}{2}$. Then f have a unique fixed point in X .

This result is **Kannan** contraction condition in complex valued b -metric space.

Corollary 7. Let (X, d) be a complete complex valued b -metric space with coefficient $s \geq 1$ and $f : X \rightarrow X$ be self-map satisfying the following condition:

$$d(fx, fy) \leq \alpha_4[d(x, fy) + d(y, fx)],$$

where each of $\alpha_4 \geq 0$ and $s\alpha_4 = s\alpha_5 < \frac{1}{2}$. Then f have a unique fixed point in X .

This result is **Chatterjea** contraction condition in complex valued b -metric space.

Corollary 8. Let (X, d) be a complete complex valued b -metric space with coefficient $s \geq 1$ and $f : X \rightarrow X$ be self-map satisfying the following condition:

$$d(fx, fy) \leq \alpha_1 d(x, y) + \alpha_2 d(x, fx) + \alpha_3 d(y, fy),$$

where each of $\alpha_i \geq 0$ and $\alpha_1 + s\alpha_2 + \alpha_3 < 1$. Then f have a unique fixed point in X .

This result is **Reich** contraction condition in complex valued b -metric space.

5. CONCLUSION

In this article we have extended Hardy and Roger's [10] result in complex-valued b -metric spaces. This result has also extended the results of **Kannan**, **Chatterjea**, **Reich**, etc. We have provided an example in support of condition used in our theorems.

Further, the obtained results scope for extension of many results available in the literature in future.

REFERENCES

- [1] S. Ali, *A Common Fixed Point Result in Complex Valued b -Metric space under Contractive Condition*, Global Journal of Pure and Applied Mathematics, 13 (9) (2017), 4869-4876.
- [2] A. Azam, F. Brain, M. Khan, *Common fixed point theorems in complex valued metric spaces*, Numerical Functional Analysis and Optimization, 32 (3) (2011), 243-253.
- [3] K. Berrah, A. Aliouche, T.E. Oussaeif, *Applications and theorem on common fixed point in complex valued b -metric space*, AIMS Mathematics, 4 (3) (2019), 1019-1033.
- [4] V.P. Bhardwaj, K. Wadhwa, *Some fuzzy common fixed point theorems using common limit in the range property with an application*, Mathematica Moravica, 24 (2) (2020), 3349.
- [5] H. Bouhadjera, *Fixed points for occasionally weakly biased mappings of type (A)*, Mathematica Moravica, 26 (1) (2022), 113-122.
- [6] S.K. Chatterjea, *Fixed point theorems*, Comptes rendus de l'Académie bulgare des Sciences, 25 (1972), 727-730.
- [7] A.K. Dubey, R. Shukla, R.P. Dubey, *Some Fixed Point Theorems in Complex Valued b -Metric Spaces*, Journal of Complex Systems, 2015 (2015), Article ID: 832467, 7 pages.

- [8] A.K. Dubey, M. Tripathi, *Common Fixed Point Theorem in Complex Valued b -metric Space for Rational Contractions*, Journal of Informatics and Mathematical Sciences, 7 (3) (2015), 149-161.
- [9] T. Hamaizia, P.P. Murthy, *Z-contraction condition involving simulation function in b -metric space under fixed points considerations*, Mathematica Moravica, 25 (2) (2021), 43-52.
- [10] G.E. Hardy, T.D. Rogers, *A generalization of a fixed point theorem of Reich*, Canadian Mathematical Bulletin, 16 (1973), 201-206.
- [11] R. Kannan, *Some results on fixed points*, Bulletin of the Calcutta Mathematical Society, 60 (1968), 71-76.
- [12] L.A. Lusternik, V.J. Sobalev, *Elements of functional analysis*, Hindustan Publishing Corporation(I), Delhi, 1974.
- [13] A.A. Mukheimer, *Some Common Fixed Point theorems in Complex Valued b -Metric spaces*, Scientific World Journal, 2014 (2014), Article ID: 587825, 6 pages.
- [14] K.P.R. Rao, P. Ranga Swamy, J.R. Prasad, *A common fixed point theorem in complex valued b -metric spaces*, Bulletin of Mathematics and statistics Research, 1 (1) (2013), 1-8.
- [15] S. Reich, *Fixed points of contractive functions*, Bollettino dell'Unione Matematica Italiana, 5 (1972), 26-42.
- [16] F. Rouzkard, M. Imdad, *Some common fixed point theorems on complex valued metric spaces*, Computers and Mathematics with Applications, 64 (6) (2012), 1866-1874.
- [17] G.S. Saluja, *Fixed point theorems for cyclic contractions in S -metric spaces involving C -class function*, Mathematica Moravica, 26 (1) (2022), 57-76.
- [18] N. Singh, D. Singh A. Badal, V. Joshi, *Fixed point theorems in complex valued metric spaces*, Journal of the Egyptian Mathematical Society, 24 (3) (2016), 402-409.
- [19] K. Sitthikul, S. Saejung, *Some fixed point theorems in complex valued metric spaces*, Fixed Point Theory and Applications, 2012 (2012), Article ID: 189, 11 pages.

DINANATH BARMAN

DEPARTMENT OF MATHEMATICS

RAIGANJ UNIVERSITY

UTTAR DINAJPUR, WEST BENGAL-733134

INDIA

E-mail address: dinanathbarman85@gmail.com

KRISHNADHAN SARKAR

DEPARTMENT OF MATHEMATICS

RANIGANJ GIRLS' COLLEGE

RANIGANJ, P. BARDHAMAN, WEST BENGAL-713358

INDIA

E-mail address: sarkarkrishnadhan@gmail.com

KALISHANKAR TIWARY

DEPARTMENT OF MATHEMATICS

RAIGANJ UNIVERSITY

UTTAR DINAJPUR, WEST BENGAL-733134

INDIA

E-mail address: tiwarykalishankar@yahoo.com