Mathematica Moravica
Vol. **27**, No. **2** (2023), 25–32 doi: 10.5937/MatMor2302025P

Blow-up phenomena for a $p(x)$-biharmonic heat equation with variable exponent

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Abstract. In this paper, we deal with a $p(x)$-biharmonic heat equation with variable exponent under Dirichlet boundary and initial condition. We prove the blow up of solutions under suitable conditions.

1. Introduction

Let $\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$. We are concerned with the following $p(x)$-biharmonic heat equation, with variable exponent, of the form

$$\text{(1)} \qquad \begin{cases} u_t + \Delta^2 u_t + \Delta^2_{p(x)} u = |u|^{q(x)-2} u, & Q = \Omega \times (0,T), \\ u(x,t) = \Delta u(x,t) = 0, & \partial Q = \partial\Omega \times [0,T), \\ u(x,0) = u_0(x), & \Omega, \end{cases}$$

where $\Delta^2_{p(x)}$ is the so-called the $p(x)$-biharmonic operator and is defined by

$$\Delta^2_{p(x)} u = \Delta \left(|\Delta u|^{p(x)-2} \Delta u \right).$$

The exponents $p(.)$ and $q(.)$ are given measurable functions on $\overline{\Omega}$ such that

$$\text{(2)} \qquad 2 \le p_- \le p(x) \le p_+ < q_- \le q(x) \le q_+ < p_*(x),$$

with

$$p_*(x) = \begin{cases} \frac{np(x)}{(n-p(x))_+}, & \text{if } p_+ < n, \\ +\infty, & \text{if } p_+ \ge n. \end{cases}$$

We also suppose that

$$\text{(3)} \qquad |p(x) - p(y)| \le \frac{A}{\log\left(\frac{1}{|x-y|}\right)}, \text{ for all } x,\ y\ \in \Omega \text{ with } |x-y| < \delta,$$

with $A > 0$, $0 < \delta < 1$ and

$$\text{(4)} \qquad ess \inf \left(p^*(x) - q(x) \right) > 0.$$

2020 *Mathematics Subject Classification.* Primary: 40B05; Secondary: 35B44, 35K05.
Key words and phrases. Blow up, heat equation, variable exponent.
Full paper. Received 9 January 2023, revised 18 August 2023, accepted 3 September 2023, available online 22 September 2023.

The following problem was considered by Alaoui et al. in [2]

$$u_t - div\left(|\nabla u|^{m(x)-2}\nabla u\right) = |u|^{p(x)-2}u. \tag{5}$$

The authors proved the blow up of solutions. Later, Rahmoune [13] proved an upper bound for blow up time of solutions eq. (5).

Di et al. [5] considered the following pseudo-parabolic equation with variable exponent

$$u_t - \Delta u_t - div\left(|\nabla u|^{m(x)-2}\nabla u\right) = |u|^{p(x)-2}u. \tag{6}$$

They proved an upper bound and lower bound for blow up time. Later, some authors studied blow up of solutions of the equation (6) (see [8, 15]).

Liu [9] studied the $p(x)$-biharmonic heat equation

$$u_t + \Delta^2_{p(x)}u = |u|^{q(x)-2}u.$$

The author proved the local existence and blow up of solutions. Ferreira et al. [7] considered the beam-equation with a strong damping and the $p(x)$-biharmonic operator

$$u_{tt} + \Delta^2_{p(x)}u - \Delta u_t + f(x,t,u_t) = g(x,t).$$

They proved the local and global existence of solutions. Some other researchers considered the parabolic-type equations with variable exponents (see [3, 10, 11]).

The problems with variable exponents arise in many branches of sciences such as electrorheological fluids, nonlinear elasticity theory and image processing [4, 6, 14].

Motivated by the above studies, in this paper, we consider the blow up of the solution (1) under some conditions.

The present paper is structured as follows. In Section 2, we state some results about the variable exponent $L^{p(x)}(\Omega)$ Lebesgue and $W^{m,p(x)}(\Omega)$ Sobolev spaces. In Section 3, the blow up phenomena will be proved.

2. Preliminaries

We recall some well-known results about the Lebesgue spaces and Sobolev spaces with variable exponents (see [6, 12]).

Let $p : \Omega \to [1,\infty]$ be a measurable function, where Ω is a bounded domain of R^n. We define the Lebesgue space with variable exponent $p(.)$ by

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to R,\ u \text{ is measurable and } \rho_{p(.)}(\lambda u) < \infty,\ \text{for some } \lambda > 0\right\},$$

where

$$\rho_{p(.)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

Also endowed with the Luxemburg-type norm

$$\|u\|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \leq 1\right\},$$

$L^{p(x)}(\Omega)$ is a Banach space.

The Sobolev space with variable exponent $W^{m,p(x)}(\Omega)$ is defined as

$$W^{m,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega),\ |\alpha| \leq m\right\}.$$

Sobolev space with variable exponent is a Banach space with respect to the norm

$$\|u\|_{2,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)} + \|\Delta u\|_{p(x)}.$$

Lemma 1 ([2]). (i) *If* (4) *holds, then* $\|u\|_{p(.)} \leq C\|\nabla u\|_{p(.)}$ *for all* $u \in W_0^{1,p(.)}(\Omega)$, *where* Ω *is bounded. In paticular, the space* $W_0^{1,p(.)}(\Omega)$ *has a norm given by* $\|u\|_{1,p(.)} = \|\nabla u\|_{p(.)}$, *for all* $u \in W_0^{1,p(.)}(\Omega)$.

(ii) *If* $p \in C(\overline{\Omega})$, $q : \Omega \to [1,\infty)$ *is a measurable function and*

$$ess\inf\left(p^*(x) - q(x)\right) > 0,$$

with $p^*(x) = \frac{np(x)}{(n-p(x))_+}$ *then*

$$W_0^{1,p(.)}(\Omega) \hookrightarrow\hookrightarrow L^{q(.)}(\Omega).$$

3. Blow up

In this part, we study blow-up result of solutions. We first state a local existence theorem [1].

Theorem 1. *For all* $u_0 \in W_0^{1,p(.)}(\Omega)$, *there exists a number* $T_0 \in (0,T]$ *such that the problem* (1) *has a strong solution* u *on* $[0,T_0]$ *satisfying*

$$u \in C_w\left([0,T_0]; W_0^{1,p(.)}(\Omega)\right) \cap C\left([0,T_0], L^{q(.)}(\Omega)\right) \cap W^{1,2}\left(0,T_0; L^2(\Omega)\right).$$

Lemma 2.

$$E(t) = \int_\Omega \left(\frac{1}{p(x)}|\Delta u|^{p(x)} - \frac{1}{q(x)}|u|^{q(x)}\right) dx$$

is a nonincreasing function for $t \geq 0$ *and*

$$E'(t) \leq 0.$$

Proof. Multiplying u_t on two sides of the equation (1), and integrating by parts, we get

$$\int_\Omega u_t^2 dx + \int_\Omega \Delta u_t^2 dx + \frac{d}{dt}\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)} dx = \frac{d}{dt}\int_\Omega \frac{1}{q(x)}|u|^{q(x)} dx.$$

We then define the energy by

$$E(t)=\int_{\Omega}\left(\frac{1}{p(x)}|\Delta u|^{p(x)}-\frac{1}{q(x)}|u|^{q(x)}\right)dx.$$

Clearly, we get

$$E'(t)=-\int_{\Omega}u_t^2dx-\int_{\Omega}|\Delta u_t|^2\,dx\le 0.$$

Let $H(t)=-E(t)$. So, $H'(t)\ge 0$. □

Theorem 2. *Let* $u_0\in W_0^{1,m(.)}(\Omega)$ *such that* $\int_{\Omega}u_0^2dx+\int_{\Omega}\Delta^2u_0dx>0$ *and*

$$\int_{\Omega}\left(\frac{1}{p(x)}|\Delta u_0|^{p(x)}-\frac{1}{q(x)}|u_0|^{q(x)}\right)dx\ge 0.$$

Then

$$F(t)=\frac{1}{2}\left(\int_{\Omega}u^2dx+\int_{\Omega}|\Delta u|^2\,dx\right)$$

blows up in finite time $t^*<+\infty$.

Proof. By differentiating F with respect to t, we obtain

$$\begin{aligned}F'(t)&=\int_{\Omega}(uu_t+\Delta u\Delta u_t)\,dx\\&=\int_{\Omega}\left[u\left(-\Delta^2u_t+div\left(|\Delta u|^{p(x)-2}\Delta u\right)+|u|^{q(x)-2}u\right)+\Delta u\Delta u_t\right]dx\\&=\int_{\Omega}\left(|u|^{q(x)}-|\Delta u|^{p(x)}\right)dx\\&=\int_{\Omega}q(x)\left(\frac{|u|^{q(x)}}{q(x)}-\frac{|\Delta u|^{p(x)}}{p(x)}\right)dx\\&\quad+\int_{\Omega}q(x)\left(\frac{1}{p(x)}-\frac{1}{q(x)}\right)|\Delta u|^{p(x)}\,dx.\end{aligned}$$

Since $E'(t)\le 0$, we get

$$\int_{\Omega}q(x)\left(\frac{|u|^{q(x)}}{q(x)}-\frac{|\Delta u|^{p(x)}}{p(x)}\right)dx\ \ge\ \int_{\Omega}q(x)\left(\frac{|u_0|^{q(x)}}{q(x)}-\frac{|\Delta u_0|^{p(x)}}{p(x)}\right)dx\ \ge$$

$$q_-\int_{\Omega}\left(\frac{|u_0|^{q(x)}}{q(x)}-\frac{|\Delta u_0|^{p(x)}}{p(x)}\right)dx\ \ge\ 0.$$

We see

$$F'(t) \geq \int_\Omega q_- \left[\frac{1}{p^+} - \frac{1}{q^-}\right] |\Delta u|^{p(x)}\, dx = C_0 \int_\Omega |\Delta u|^{p(x)}.$$

We define the sets $\Omega_+ = \{x \in \Omega : |\Delta u| \geq 1\}$ and $\Omega_- = \{x \in \Omega : |\Delta u| < 1\}$. So,

$$\begin{aligned} F'(t) &\geq C_0 \Big(\int_{\Omega_-} |\Delta u|^{p_+} + \int_{\Omega_+} |\Delta u|^{p_-} \Big) \\ &\geq C_1 \left(\Big(\int_{\Omega_-} |\Delta u|^2\, dx \Big)^{p_+/2} + \Big(\int_{\Omega_+} |\Delta u|^2\, dx \Big)^{p_-/2} \right), \end{aligned}$$

using the fact that $\|\Delta u\|_2 \leq C\, \|\Delta u\|_r$, for all $r \geq 2$.

This implies that

$$(F'(t))^{2/p_+} \geq C_2 \int_{\Omega_-} |\Delta u|^2\, dx$$

and similarly,

$$(F'(t))^{2/p_-} \geq C_3 \int_{\Omega_+} |\Delta u|^2\, dx.$$

The Poincare inequality gives $\|\Delta u\|^2 \geq \lambda_1 \|u\|^2$, where λ_1 is the first eigenvalue of the problem

$$\begin{cases} \Delta^2 w + \lambda w = 0, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases}$$

Thus, we obtain

$$\begin{aligned} \|\Delta u\|^2 &= \frac{1}{1+\lambda_1} \|\Delta u\|^2 + \frac{\lambda_1}{1+\lambda_1} \|\Delta u\|^2 \\ &\geq \frac{\lambda_1}{1+\lambda_1} \|u\|^2_{H_0^2}. \end{aligned}$$

Simple addition leads to

$$\begin{aligned} (F'(t))^{2/p_-} + (F'(t))^{2/p_+} &\geq (C_3 + C_2)\, \|\Delta u\|^2 \\ &\geq \frac{\lambda_1 (C_3 + C_2)}{1+\lambda_1} \|u\|^2_{H_0^2} = C_4 F(t), \end{aligned} \tag{7}$$

or

$$(8) \qquad (F'(t))^{2/p_-}\left(1+(F'(t))^{2(\frac{1}{p_+}-\frac{1}{p_-})}\right) \;\geq\; C_4F(t).$$

By (7) and the fact that $F(t) \geq F(0) > 0(F'(t) \geq 0)$, we have, for each $t > 0$, either

$$(F'(t))^{2/p_-} \;\geq\; \frac{C_4}{2}F(t) \;\geq\; \frac{C_4}{2}F(0)$$

or

$$(F'(t))^{2/p_+} \;\geq\; \frac{C_4}{2}F(t) \;\geq\; \frac{C_4}{2}F(0),$$

which gives in turn

$$F'(t) \;\geq\; C_5(F(0))^{p_-/2}$$

or

$$F'(t) \;\geq\; C_6(F(0))^{p_+/2}.$$

Hence $F'(t) \geq \alpha$, where $\alpha = \min\{C_5(F(0))^{p_-/2}, C_6(F(0))^{p_+/2}\}$.
Since $\frac{1}{p_+} - \frac{1}{p_-} \leq 0$, (8) yields

$$\big(F'(t)\big)^{2/p_-}(1+\alpha)^{2(\frac{1}{p_+}-\frac{1}{p_-})} \;\geq\; C_4F(t), \quad \forall t \geq 0.$$

Consequently,

$$(9) \qquad F'(t) \;\geq\; \beta F^{p_-/2}(t), \quad \forall t \geq 0.$$

A simple integration of (9) over $(0, t)$ then yields

$$F(t)^{1-\frac{p_-}{2}} \;\leq\; F(0)^{1-\frac{p_-}{2}} - \frac{p_- - 2}{2}\beta t,$$

which implies that

$$F(t) \;\geq\; \frac{1}{\left(F(0)^{1-\frac{p_-}{2}} - \frac{p_- -2}{2}\beta t\right)^{\frac{2}{p_- -2}}}.$$

This shows that F blows up in a time

$$t^* \;\leq\; \frac{2F(0)^{1-\frac{p_-}{2}}}{(p_- - 2)\beta}. \qquad \square$$

4. Conclusion

In this paper, we have studied a $p(x)$-biharmonic heat equation with a variable. The blow up of solutions has been proved. Our result improves earlier results in the literature.

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