Mathematica Moravica
Vol. **28**, No. **1** (2024), 97–108

Some new results for the generalized bivariate Fibonacci and Lucas polynomials

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Abstract. In this paper, new identities are obtained by using the generalized bivariate Fibonacci and Lucas polynomials. Firstly, several binomial summations and the closed formulas for summation of powers are investigated for these polynomials. Also, general summation formulas, different generating functions, and relations of these polynomials are presented.

1. Introduction

As well-studied objects in mathematics, the Fibonacci, Lucas, Pell and Chebyshev numbers possess many kinds of generalizations. One of the most important generalizations is the Fibonacci polynomial [1–10]. Due to common usage of this polynomial in applied sciences, some generalizations have been defined in the literature [13, 15, 19, 20]. Some recent work in this direction can be seen in [21–24, 26]. In [12] and its references, a short history and comprehensive information about the Fibonacci polynomial can be found. The Fibonacci numbers are defined as

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, \quad F_1 = 1 \tag{1}$$

for $n \geq 2$. In [14], the authors gave a new generalization of the Fibonacci and Lucas polynomials which are called $h(x)$-Fibonacci and $h(x)$-Lucas polynomials. Then, the authors presented the several studies on generalization of Fibonacci numbers in [16–18]. Also, in [25], the author investigated some arithmetic properties for the (p, q)-Fibonacci and Lucas polynomials associated with the classical Fibonacci and Lucas numbers. In [11], the authors gave a new generalization of the Fibonacci and Lucas polynomials which are called generalized bivariate Fibonacci and Lucas polynomials.

2020 *Mathematics Subject Classification.* Primary: 11B39.
Key words and phrases. Binomial, Fibonacci polynomials, generating function, Lucas polynomials.
Full paper. Received 5 September 2023, accepted 31 May 2024, available online 17 June 2024.

For $n \geq 2$ and $p(x,y)$, $q(x,y)$ polynomials with real coefficients, the generalized bivariate Fibonacci and Lucas polynomials are described by

$$H_n(x,y) = p(x,y)H_{n-1}(x,y) + q(x,y)H_{n-2}(x,y), \tag{2}$$

$$K_n(x,y) = p(x,y)K_{n-1}(x,y) + q(x,y)K_{n-2}(x,y), \tag{3}$$

where $H_0(x,y) = 0$, $H_1(x,y) = 1$, $K_0(x,y) = 2$, $K_1(x,y) = p(x,y)$ and $p^2(x,y) + 4q(x,y) > 0$. The Binet formulas, relation of generalized bivariate Fibonacci and Lucas polynomials are (see[11]):

$$H_n(x,y) = \frac{\alpha^n(x,y) - (-q(x,y))^n \alpha^{-n}(x,y)}{\alpha(x,y) + q(x,y)\alpha^{-1}(x,y)}, \tag{4}$$

$$K_n(x,y) = \alpha^n(x,y) + (-q(x,y))^n \alpha^{-n}(x,y), \tag{5}$$

$$K_n(x,y) = H_{n+1}(x,y) + q(x,y)H_{n-1}(x,y), \tag{6}$$

where $\alpha(x,y)$ and $\beta(x,y) = -q(x,y)\alpha^{-1}(x,y)$ are roots of characteristic equations (2) and (3).

For different $p(x,y)$ and $q(x,y)$, we obtain different polynomial sequences by using recursive relation. These polynomial sequences are given in the Table 1.

Table 1. Special conditions of the generalized bivariate Fibonacci and Lucas polynomials (see [11]).

$p(x,y)$	$q(x,y)$	$H_n(x,y)$	$K_n(x,y)$
x	y	Bivariate Fibonacci, $F_n(x,y)$	Bivariate Lucas, $L_n(x,y)$
x	1	Fibonacci, $F_n(x)$	Lucas, $L_n(x)$
$2x$	1	Pell, $P_n(x)$	Pell-Lucas, $Q_n(x)$
1	$2x$	Jacobsthal, $J_n(x)$	Jacobsthal-Lucas, $j_n(x)$
$2x$	-1	Chebyshev of the second kind, $U_{n-1}(x)$	Chebyshev of the first kind, $2T_n(x)$
$3x$	-2	Fermat, $\mathcal{F}_n(x)$	Fermat-Lucas, $\mathcal{F}_n(x)$

We would like to mention hereafter that, we use notations $\alpha = \alpha(x,y)$, $\beta = \beta(x,y)$ and $p = p(x,y)$, $q = q(x,y)$ for brevity.

Motivated by the work of [11], we wanted to extend the results on the generalized bivariate Fibonacci and Lucas polynomials. In Section 2, we provide properties for the generalized bivariate Fibonacci polynomials. In Section 3, we obtain identities for generalized bivariate Lucas polynomials that generalize the Lucas polynomials.

2. On properties of generalized bivariate Fibonacci polynomials

We obtain different binomial summations of the generalized bivariate Fibonacci polynomials in the following theorems.

Theorem 1. *For $n \geq 1$ and $k \geq 0$ integers, we get*

(i) $\sum_{i=0}^{n} \binom{n}{i} (-q)^{ik} H_{(n-2i)k}(x,y) = 0;$

(ii) $\sum_{i=0}^{n} \binom{n}{i} (-1)^i (-q)^{ik} H_{(n-2i)k}(x,y) =$

$$\begin{cases} 2(p^2+4q)^{\frac{n-1}{2}} H_k^n(x,y), & n \text{ is odd;} \\ 0, & n \text{ is even.} \end{cases}$$

(iii) $\sum_{i=0}^{n} \binom{n}{i} (-q)^{(n-i)k} H_{ik}^2(x,y) = \dfrac{K_{nk}(x,y) K_k^n(x,y) - 2^{n+1}(-q)^{nk}}{p^2+4q}.$

Proof. (i) From the equation (4), we write

$$\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} (-q)^{ik} H_{(n-2i)k}(x,y) \\
&= \sum_{i=0}^{n} \binom{n}{i} (-q)^{ik} \left(\frac{\alpha^{(n-2i)k} - \beta^{(n-2i)k}}{\alpha - \beta} \right) \\
&= \frac{1}{\alpha-\beta} \sum_{i=0}^{n} \binom{n}{i} \alpha^{nk-ik} \beta^{ik} - \frac{1}{\alpha-\beta} \sum_{i=0}^{n} \binom{n}{i} \alpha^{ik} \beta^{nk-ik} \\
&= 0.
\end{aligned}$$

(ii) Again, from the equation (4), we have

$$\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} (-1)^i (-q)^{ik} H_{(n-2i)k}(x,y) \\
&= \sum_{i=0}^{n} \binom{n}{i} (-1)^i (-q)^{ik} \left(\frac{\alpha^{(n-2i)k} - \beta^{(n-2i)k}}{\alpha - \beta} \right) \\
&= \frac{1}{\alpha-\beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \alpha^{nk-ik} \beta^{ik} - \frac{1}{\alpha-\beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \alpha^{ik} \beta^{nk-ik} \\
&= \begin{cases} 2(p^2+4q)^{\frac{n-1}{2}} H_k^n(x,y), & n \text{ is odd;} \\ 0, & n \text{ is even.} \end{cases}
\end{aligned}$$

(iii) Similarly to (i) and (ii), from the equation (4), we have

$$\sum_{i=0}^{n}\binom{n}{i}(-q)^{(n-i)k}H_{ik}^{2}(x,y)=\sum_{i=0}^{n}\binom{n}{i}(-q)^{(n-i)k}\left(\frac{\alpha^{ik}-\beta^{ik}}{\alpha-\beta}\right)^{2}$$
$$=\frac{1}{(\alpha-\beta)^{2}}\sum_{i=0}^{n}\binom{n}{i}\alpha^{nk+ik}\beta^{nk-ik}-\frac{1}{(\alpha-\beta)^{2}}\sum_{i=0}^{n}\binom{n}{i}2(-q)^{nk}$$
$$+\frac{1}{(\alpha-\beta)^{2}}\sum_{i=0}^{n}\binom{n}{i}\alpha^{nk-ik}\beta^{nk+ik}$$
$$=\frac{K_{nk}(x,y)K_{k}^{n}(x,y)-2^{n+1}(-q)^{nk}}{p^{2}+4q}.$$
□

Theorem 2. *The identities are hold:*

(i) $\sum_{k=0}^{n}\binom{n}{k}p^{k}q^{n-k}H_{k}(x,y)=H_{2n}(x,y)$,

(ii) $\sum_{k=0}^{n}\binom{n}{k}p^{k}q^{n-k}H_{k+1}(x,y)=H_{2n+1}(x,y)$.

Proof. (i) By using the equation (4), we get

$$\sum_{k=0}^{n}\binom{n}{k}p^{k}q^{n-k}H_{k}(x,y)=\sum_{k=0}^{n}\binom{n}{k}p^{k}q^{n-k}\left(\frac{\alpha^{k}-\beta^{k}}{\alpha-\beta}\right)$$
$$=\frac{q^{n}}{\alpha-\beta}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{p}{q}\right)^{k}\alpha^{k}-\frac{q^{n}}{\alpha-\beta}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{p}{q}\right)^{k}\beta^{k}$$
$$=\frac{1}{\alpha-\beta}\left(p\alpha+q\right)^{n}-\frac{1}{\alpha-\beta}\left(p\beta+q\right)^{n}.$$

By considering well-known equalities $p\alpha+q=\alpha^{2}$, $p\beta+q=\beta^{2}$ and again the equation (4), we obtain claimed result.

(ii) By taking into account the equation (4) and above operations, we achieve the desired result. □

We take account Theorem 1 to get the formula of sums of any positive powers for the generalized bivariate Fibonacci polynomials.

Theorem 3. *For $m,n\geq 1$ integers, we have*

$$\sum_{k=0}^{m}H_{k}^{n}(x,y)=$$
$$\begin{cases}\dfrac{1}{2(p^{2}+4q)^{\frac{n-1}{2}}}\displaystyle\sum_{i=0}^{n}\binom{n}{i}\frac{(-1)^{i}C}{(-q)^{n}-(-q)^{i}K_{n-2i}(x,y)+1}, & n \text{ is odd;}\\ \dfrac{1}{2(p^{2}+4q)^{\frac{n}{2}}}\displaystyle\sum_{i=0}^{n}\binom{n}{i}\frac{(-1)^{i}D}{(-q)^{n}-(-q)^{i}K_{n-2i}(x,y)+1}, & n \text{ is even;}\end{cases}$$

where

$$C = (-q)^{mi+n} H_{(n-2i)m}(x, y) - (-q)^{i(m+1)} H_{(n-2i)(m+1)}(x, y) + (-q)^i H_{n-2i}(x, y),$$

$$D = (-q)^{mi+n} K_{(n-2i)m}(x, y) - (-q)^{i(m+1)} K_{(n-2i)(m+1)}(x, y) - (-q)^i K_{n-2i}(x, y) + 2.$$

Proof. There are two condition for n. If n is odd, from the equation (4) and Theorem 1, we write

$$\begin{aligned}\sum_{k=0}^{m} H_k^n(x, y) &= \frac{1}{2(p^2+4q)^{\frac{n-1}{2}}} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \sum_{k=0}^{m} (-q)^{ik} H_{(n-2i)k}(x, y) \\ &= \frac{1}{2(p^2+4q)^{\frac{n+1}{2}}} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \frac{(-q)^{i(m+1)} \alpha^{(n-2i)(m+1)} - 1}{(-q)^i \alpha^{n-2i} - 1} \\ &\quad - \frac{1}{2(p^2+4q)^{\frac{n+1}{2}}} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \frac{(-q)^{i(m+1)} \beta^{(n-2i)(m+1)} - 1}{(-q)^i \beta^{n-2i} - 1}.\end{aligned}$$

By considering a well-known equality $\alpha\beta = -q$, the equations (4) and (5), we obtain

$$\sum_{k=0}^{m} H_k^n(x, y) = \frac{1}{2(p^2+4q)^{\frac{n-1}{2}}} \sum_{i=0}^{n} \binom{n}{i} \frac{(-1)^i C}{(-q)^n - (-q)^i K_{n-2i}(x, y) + 1}.$$

As secondly, for n even, it can be similarly proved. □

We obtain general summations of the generalized bivariate Fibonacci polynomials in the following theorem.

Theorem 4. *For $n \geq 1$ and $j, m \geq 0$ integers, we have*

$$\sum_{k=0}^{n-1} H_{mk+j}(x, y) =$$

$$\begin{cases} \dfrac{(-q)^m H_{mn+j-m}(x, y) - H_{mn+j}(x, y) + (-q)^j H_{m-j}(x, y) + H_j(x, y)}{(-q)^m - K_m(x, y) + 1}, & j < m; \\ \dfrac{(-q)^m H_{mn+j-m}(x, y) - H_{mn+j}(x, y) - (-q)^m H_{j-m}(x, y) + H_j(x, y)}{(-q)^m - K_m(x, y) + 1}, & j \geq m; \end{cases}$$

where $(-q)^m - K_m(x, y) + 1 \neq 0$.

Proof. From the equation (4), we write

$$\begin{aligned}\sum_{k=0}^{n-1} H_{mk+j}(x, y) &= \sum_{k=0}^{n-1} \frac{\alpha^{mk+j} - \beta^{mk+j}}{\alpha - \beta} \\ &= \frac{1}{\alpha - \beta} \left[\alpha^j \frac{\alpha^{mn} - 1}{\alpha^m - 1} - \beta^j \frac{\beta^{mn} - 1}{\beta^m - 1} \right].\end{aligned}$$

By considering q well-known equality $\alpha\beta = -q$ and the equation (5), we get

$$\sum_{k=0}^{n-1} H_{mk+j}(x,y) = \frac{1}{\alpha-\beta}\left[\frac{(-q)^m\alpha^{mn+j-m} - \alpha^{mn+j} - \alpha^j\beta^m + \alpha^j}{(-q)^m - K_m(x,y) + 1}\right] - \frac{1}{\alpha-\beta}\left[\frac{(-q)^m\beta^{mn+j-m} - \beta^{mn+j} - \alpha^m\beta^j + \beta^j}{(-q)^m - K_m(x,y) + 1}\right].$$

Again, from equations (4) and (5) we obtain the desired result. □

If we take $m = 1, j = 0$ in Theorem 4, we obtain the following results, also available in [11].

Corollary 1. *For $n \geq 1$ integers, we obtain*

$$\sum_{k=0}^{n-1} H_k(x,y) = \frac{H_n(x,y) + qH_{n-1}(x,y) - 1}{p+q-1},$$

where $p + q - 1 \neq 0$.

If we take $m = 2$, $j = 0$ in Theorem 4, the result is as follows.

Corollary 2. *For $n \geq 1$ integers, we have*

$$\sum_{k=0}^{n-1} H_{2k}(x,y) = \frac{H_{2n}(x,y) - q^2 H_{2n-2}(x,y) - p}{p^2 - q^2 + 2q - 1},$$

where $p^2 - q^2 + 2q - 1 \neq 0$.

If we take $m = 2$, $j = 1$ in Theorem 4, the result is as follows.

Corollary 3. *For $n \geq 1$ integers, we get*

$$\sum_{k=0}^{n-1} H_{2k+1}(x,y) = \frac{H_{2n+1}(x,y) - q^2 H_{2n-1}(x,y) + q - 1}{p^2 - q^2 + 2q - 1},$$

where $p^2 - q^2 + 2q - 1 \neq 0$.

We obtain the exponential and Poisson generating functions of the generalized bivariate Fibonacci polynomials in the following theorem.

Theorem 5. *The identities are hold:*

(i) $$\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} = \frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta},$$

(ii) $$\sum_{n=0}^{\infty} H_n(x,y)\frac{e^{-t}t^n}{n!} = \frac{e^{(\alpha-1)t} - e^{(\beta-1)t}}{\alpha - \beta}.$$

Proof. (i) From the equation (4) and the MacLaurin expansion for the exponential function, we have

$$\begin{aligned}\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)\frac{t^n}{n!}\\ &= \frac{1}{\alpha-\beta}\sum_{n=0}^{\infty}\frac{\alpha^n t^n}{n!} - \frac{1}{\alpha-\beta}\sum_{n=0}^{\infty}\frac{\beta^n t^n}{n!}\\ &= \frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}.\end{aligned}$$

The proof for (ii) can be done in a similar way to (i). □

Further, we present the Vajda identity for generalized bivariate Fibonacci polynomials.

Theorem 6. *For n, r, s integers, we have*

$$H_{n+r}(x,y)H_{n+s}(x,y) - H_n(x,y)H_{n+r+s}(x,y) = (-q)^n H_s(x,y)H_r(x,y)$$

Proof. Using the equations (4), for the left hand side LHS, we get

$$\begin{aligned}LHS &= \left(\frac{\alpha^{n+r}-\beta^{n+r}}{\alpha-\beta}\right)\left(\frac{\alpha^{n+s}-\beta^{n+s}}{\alpha-\beta}\right)\\ &\quad - \left(\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}\right)\left(\frac{\alpha^{n+r+s}-\beta^{n+r+s}}{\alpha-\beta}\right)\\ &= (-q)^n\left(\frac{\alpha^{s}-\beta^{s}}{\alpha-\beta}\right)\left(\frac{\alpha^{r}-\beta^{r}}{\alpha-\beta}\right)\\ &= (-q)^n H_s(x,y)H_r(x,y).\end{aligned}$$ □

By taking $s = -r = 1$ in Theorem 6, we obtain the Cassini identity for generalized bivariate Fibonacci polynomials as Theorem 8 in [11].

Corollary 4. *For n integers, we have*

$$H_{n-1}(x,y)H_{n+1}(x,y) - H_n^2(x,y) = -(-q)^{n-1}.$$

By taking $s = m-n, r = 1$ in Theorem 6, we obtain the D'Ocagne identity for generalized bivariate Fibonacci polynomials as Theorem 4 in [11].

Corollary 5. *For n, m integers, we have*

$$H_{n+1}(x,y)H_m(x,y) - H_n(x,y)H_{m+1}(x,y) = (-q)^n H_{m-n}(x,y).$$

By taking $r = -s$ in Theorem 6, we obtain the Catalan identity for generalized bivariate Fibonacci polynomials as Theorem 3 in [11].

Corollary 6. *For n, s integers, we have*

$$H_{n-s}(x,y)H_{n+s}(x,y) - H_n^2(x,y) = -(-q)^{n-s}H_s^2(x,y).$$

The following theorem gives us the relationships between the generalized bivariate Fibonacci and Lucas polynomials.

Theorem 7. *For $n \geq 2$ integers, we have*

(i) $H_{n+2}(x,y) - qH_{n-2}(x,y) = pK_n(x,y)$,

(ii) $(p^2+4q)H_n(x,y) + pK_n(x,y) = 2K_{n+1}(x,y)$.

Proof. (i) We get together with the equations (2), (3) and (6)

$$\begin{aligned} H_{n+2}(x,y) - qH_{n-2}(x,y) &= pH_{n+1}(x,y) + pqH_{n-1}(x,y) \\ &= pK_n(x,y). \end{aligned}$$

Since the proof of (ii) is same as the proof of (i), we omit these proofs to cut the unnecessary repetition. □

3. ON PROPERTIES OF GENERALIZED BIVARIATE LUCAS POLYNOMIALS

In this section, we investigated the generalized bivariate Lucas polynomials. We do not write down all proofs, since they are similar to those in Section 2.

We obtain the different binomial summations of the generalized bivariate Lucas polynomials in the following theorems.

Theorem 8. *For $n \geq 1$ and $k \geq 0$ integers, we get*

(i) $$\sum_{i=0}^{n} \binom{n}{i} (-q)^{ik} K_{(n-2i)k}(x,y) = 2K_k^n(x,y),$$

(ii) $$\sum_{i=0}^{n} \binom{n}{i} (-1)^i (-q)^{ik} K_{(n-2i)k}(x,y) = \begin{cases} 2(p^2+4q)^{\frac{n}{2}} H_k^n(x,y), & n \text{ is even;} \\ 0, & n \text{ is odd;} \end{cases}$$

(iii) $$\sum_{i=0}^{n} \binom{n}{i} (-q)^{(n-i)k} K_{ik}^2(x,y) = K_{nk}(x,y)K_k^n(x,y) + 2^{n+1}(-q)^{nk},$$

(iv) $$\sum_{i=0}^{n} \binom{n}{i} (-q)^{(n-i)k} H_{ik}(x,y)K_{ik}(x,y) = H_{nk}(x,y)K_k^n(x,y).$$

Theorem 9. *The following identities hold:*

(i) $$\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} K_k(x,y) = K_{2n}(x,y),$$

(ii) $$\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} K_{k+1}(x,y) = K_{2n+1}(x,y).$$

We take intoaccount Theorem 8 to get the formula of sums of any positive powers for the generalized bivariate Lucas polynomials.

Theorem 10. *For $m, n \geq 1$ integers, we have*

$$\sum_{k=0}^{m} K_k^n(x,y) = \frac{1}{2}\sum_{i=0}^{n}\binom{n}{i}\frac{D}{(-q)^n - (-q)^i K_{n-2i}(x,y) + 1},$$

where

$$D = (-q)^{mi+n} K_{(n-2i)m}(x,y) - (-q)^{i(m+1)} K_{(n-2i)(m+1)}(x,y) - (-q)^i K_{n-2i}(x,y) + 2.$$

We obtain general summations of the generalized bivariate Lucas polynomials in the following theorem.

Theorem 11. *For $n \geq 1$ and $m, j \geq 0$ integers, we have*

$$\sum_{k=0}^{n-1} K_{mk+j}(x,y) =$$
$$\begin{cases} \dfrac{(-q)^m K_{mn+j-m}(x,y) - K_{mn+j}(x,y) - (-q)^j K_{m-j}(x,y) + K_j(x,y)}{(-q)^m - K_m(x,y) + 1}, & j < m; \\ \dfrac{(-q)^m K_{mn+j-m}(x,y) - K_{mn+j}(x,y) - (-q)^m K_{j-m}(x,y) + K_j(x,y)}{(-q)^m - K_m(x,y) + 1}, & j \geq m; \end{cases}$$

where $(-q)^m - K_m(x,y) + 1 \neq 0$.

If we take $m = 1, j = 0$ in Theorem 11, we obtain the following results, also available in [11, 21].

Corollary 7. *For $n \geq 1$ integers, we have*

$$\sum_{k=0}^{n-1} K_k(x,y) = \frac{K_n(x,y) + qK_{n-1}(x,y) + p - 2}{p + q - 1},$$

where $p + q - 1 \neq 0$.

If we take $m = 2$, $j = 0$ in Theorem 11, the results are as follows.

Corollary 8. *For $n \geq 1$ integers, we have*

$$\sum_{k=0}^{n-1} K_{2k}(x,y) = \frac{K_{2n}(x,y) - q^2 K_{2n-2}(x,y) + p^2 + 2q - 2}{p^2 - q^2 + 2q - 1},$$

where $p^2 - q^2 + 2q - 1 \neq 0$.

If we take $m = 2$, $j = 1$ in Theorem 11, the results are as follows.

Corollary 9. *For $n \geq 1$ integers, we have*

$$\sum_{k=0}^{n-1} K_{2k+1}(x,y) = \frac{K_{2n+1}(x,y) - q^2 K_{2n-1}(x,y) - pq - p}{p^2 - q^2 + 2q - 1},$$

where $p^2 - q^2 + 2q - 1 \neq 0$.

We obtain the exponential and Poisson generating functions of the generalized bivariate Lucas polynomials in the following theorem.

Theorem 12. *The following identities hold:*

(i) $\displaystyle\sum_{n=0}^{\infty} K_n(x,y) \frac{t^n}{n!} = e^{\alpha t} + e^{\beta t}$,

(ii) $\displaystyle\sum_{n=0}^{\infty} K_n(x,y) \frac{e^{-t} t^n}{n!} = e^{(\alpha-1)t} + e^{(\beta-1)t}$.

We present the Vajda identity for generalized bivariate Lucas polynomials.

Theorem 13. *For n, r, s integers, we have*

$$K_{n+r}(x,y)K_{n+s}(x,y) - K_n(x,y)K_{n+r+s}(x,y) = \\ -(-q)^n (p^2 + 4q) H_s(x,y) H_r(x,y).$$

By taking $s = -r = 1$ in Theorem 13, we obtain the Cassini identity for generalized bivariate Lucas polynomials as Theorem 4.5 in [21].

Corollary 10. *For n integers, we have*

$$K_{n-1}(x,y)K_{n+1}(x,y) - K_n^2(x,y) = (-q)^{n-1}(p^2 + 4q).$$

By taking $s = m - n, r = 1$ in Theorem 13, we obtain the D'Ocagne identity for generalized bivariate Lucas polynomials as Theorem 4.7 in [21].

Corollary 11. *For n, m integers, we have*

$$K_{n+1}(x,y)K_m(x,y) - K_n(x,y)K_{m+1}(x,y) = -(-q)^n (p^2 + 4q) H_{m-n}(x,y).$$

By taking $r = -s$ in Theorem 13, we obtain the Catalan identity for generalized bivariate Lucas polynomials as Theorem 4.6 in [21].

Corollary 12. *For n, s integers, we have*

$$K_{n-s}(x,y)K_{n+s}(x,y) - K_n^2(x,y) = (-q)^{n-s}(p^2 + 4q) H_s^2(x,y).$$

We obtain the relationships between generalized bivariate Fibonacci and Lucas polynomials.

Theorem 14. *For $n \geq 0$ and $k \geq n$ integers, we get*

(i) $K_n(x,y)H_k(x,y) - (-q)^n H_{k-n}(x,y) = H_{n+k}(x,y)$,

(ii) $K_n(x,y)K_k(x,y) - (-q)^n K_{k-n}(x,y) = K_{n+k}(x,y)$.

If we take $k = n$ in Theorem 14, we obtain the results in [11], as follows.

Corollary 13. *For $n \geq 0$ integer, we have*

(i) $K_n(x,y)H_n(x,y) = H_{2n}(x,y)$,

(ii) $K_n^2(x,y) - 2(-q)^n = K_{2n}(x,y)$.

4. Conclusion

In this paper, the generalized bivariate Fibonacci and Lucas polynomials have been investigated. Many of the properties of these polynomials are proved by the fundamental algebraic operations. Actually, the results presented here have the potential to motivate further studies of the subject of the generalized bivariate Fibonacci and Lucas polynomials with negative indices or the bivariate Horadam polynomials including bivariate Fibonacci and Lucas polynomials.

Funding

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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