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# $\mathcal{I}$-asymptotically lacunary statistical equivalent sequences in partial metric spaces

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Abstract. The present study deals with $\mathcal{I}$-asymptotically equivalent sequences in partial metric spaces. We define the notions of strongly $\mathcal{I}$-asymptotically lacunary equivalent, $\mathcal{I}$-asymptotically statistical equivalent, and $\mathcal{I}$-asymptotically lacunary statistical equivalent. We theoretically contribute to these notions and investigate some of their basic properties.

## 1. Introduction

Convergence is one of the fundamental concepts of Analysis and Functional Analysis. Statistical convergence is a generalization of the convergence and is based on the natural density of positive integers, which is essential in summability. Since Fast [1] and Steinhaus [2] examined the statistical convergence, applications and some generalizations of this concept have been given by many researchers [3–7].

Convergence of a sequence was extended to statistical convergence as follows: A sequence $(\xi_k)$ is referred to as statistically convergent to $\xi_0$ if, for all $\varepsilon > 0$,

$$\lim_n \frac{1}{n}\left|\{k \le n : |\xi_k - \xi_0| \ge \varepsilon\}\right| = 0,$$

where the $|\ \ |$ denotes the cardinality of the set $\{k \le n : |\xi_k - \xi_0| \ge \varepsilon\}$. We abbreviate it as $st - \lim \xi_k = \xi_0$. Statistical convergence for sequences has been established using natural density [8]. A natural density for the set $K$, where $K \subseteq \mathbb{N}$, has been given by

$$\delta(K) = \lim_{n\to\infty} \frac{|\{k \le n : k \in A\}|}{n}.$$

The new concept of convergence is explained by replacing the set $\{k : k < n\}$ along with a set $\{k : k_{r-1} \le k \le k_r\}$ for some lacunary sequence $(k_r)$ by Fridy and Orhan [9]. The authors compared this new convergence method

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with other summability methods and considered certain questions about the uniqueness of limit value. In the lacunary sequence $\theta = (k_r)$, we take terms with $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ when $r \to \infty$. Additionally, the interval $I_r = (k_{r-1}, k_r]$ are determined by $\theta = (k_r)$, and its term ratio $k_r$, $k_{r-1}$ is expressed as $q_r$, i.e., $q_r = \frac{k_r}{k_{r-1}}$. Lacunary statistical convergence is one of our study's primary themes. Fridy and Orhan [9] introduced lacunary statistical convergence. The sequence $\xi = (\xi_k)$ is a lacunary statistical convergent to $\xi_0$ if, for all $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r} \big| \{k \in I_r : |\xi_k - \xi_0| \geq \varepsilon\} \big| = 0.$$

It is convenient to represent symbolically by $S_\theta - \lim \xi_k = \xi_0$. The notation $S_\theta$ represents the collection of all lacunary statistical convergent sequences.

The definition of an asymptotically regular matrix that preserves the asymptotically equivalent of two non-negative sequences was given by Pobyvanets [10] in 1980 to compare the convergence rates of the sequences. However, due to the zero-valued terms of the sequences, in most cases, it was not possible to compare the terms in $\frac{x}{y}$ form. Therefore, Fridy [11] introduced new methods for comparing convergence rates. Following the work of Fridy [11], Marouf [12] investigated some necessary and sufficient conditions for a matrix to be asymptotically regular. Li [13] studied the asymptotically equivalent and summability of sequences.

In [12], two non-negative sequences $\xi = (\xi_k)$ and $\eta = (\eta_k)$ are called asymptotically equivalent, if

$$\lim_k \frac{\xi_k}{\eta_k} = 1,$$

and this is denoted by $\xi \sim \eta$. In 2003, Patterson [14] introduced asymptotically statistical equivalent by combining asymptotically equivalent and statistical convergence and extended these notions by providing statistical correspondences to the theorems. In Patterson's study, let $\xi = (\xi_k)$ and $\eta = (\eta_k)$ are two non-negative sequences. These sequences are called asymptotically statistical equivalent of multiple $\lambda$ if, for all $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{\xi_k}{\eta_k} - \lambda \right| \geq \varepsilon \right\} \right| = 0,$$

and this is denoted by $\xi \overset{S^\lambda}{\sim} \eta$. Simply asymptotically statistical equivalent if $\lambda = 1$. Moreover, let $S^\lambda$ convenient to represent symbolically the set of $\xi$ and $\eta$ such that $\xi \overset{S^\lambda}{\sim} \eta$.

Patterson and Savaş [15] conceptualized the asymptotically lacunary statistical equivalent as encompassing asymptotically equivalent, statistical convergence, and lacunary sequences.

For given a lacunary sequence $\theta = (k_r)$, and consider two non-negative sequences $\xi = (\xi_k)$ and $\eta = (\eta_k)$. These sequences are called asymptotically lacunary statistically equivalent the multiple $\lambda$ if, for all $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{\xi_k}{\eta_k} - \lambda \right| \geq \varepsilon \right\} \right| = 0.$$

This relationship is denoted as $\xi \overset{S_\theta^\lambda}{\sim} \eta$ and referred to as simply asymptotically lacunary statistically equivalent when $\lambda = 1$. Moreover, let $S_\theta^\lambda$ convenient to represent symbolically the set of $\xi$ and $\eta$ such that $\xi \overset{S_\theta^\lambda}{\sim} \eta$.

Savaş [16] investigated the notion of $\mathcal{I}$-asymptotically statistical equivalent with the respect of the multiple $\lambda$ if, for all $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ n \in N : \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{\xi_k}{\eta_k} - \lambda \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}.$$

Assume that $\theta$ is a lacunary sequence, and consider two non-negative sequences $\xi = (\xi_k)$ and $\eta = (\eta_k)$. These sequences are called strong asymptotically lacunary statistically equivalent the multiple $\lambda$ if, for all $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{\xi_k}{\eta_k} - \lambda \right| = 0.$$

This situation is denoted by $\xi \overset{N_\theta^\lambda}{\sim} \eta$ and simply strong asymptotically lacunary equivalent if $\lambda = 1$. Furthermore, let $N_\theta^\lambda$ convenient to represent symbolically the set of $\xi$ and $\eta$ such that $\xi \overset{N_\theta^\lambda}{\sim} \eta$.

In 1994, the notion of partial metric space was introduced by Matthews [17]. Unlike the metric in the usual sense, the partial metric includes the concept of a set whose distance to itself is different from zero. In this context, partial metric is a broader concept. For further details on partial metric spaces, we refer to [18,19] and many others. Convergence and summability in partial metric spaces have recently increased in popularity. Nuray [20] introduced the notions of statistical convergence and strong Cesaro summability in the mentioned spaces and investigated the relations between statistical convergence and strong Cesaro summability. Gülle et al. [21] defined the concept of ideal convergence, which is a generalization of ordinary and statistical convergence and deals with relations between newly comprehensive concepts. As a result, studies on generalized convergence concepts in partial metric spaces maintain their popularity, and relevant theories are being developed.

Let's recall the partial metric space and its related comprehensive notions. In the rest of the study, the set of non-negative real numbers will be denoted by $\mathbb{R}^{\geq 0}$.

**Definition 1** ([17]). Let $\mathbb{V}$ be a non-empty set. A function $\mathtt{p} : \mathbb{V}\times\mathbb{V} \to \mathbb{R}^{\geq 0}$ is said to be a partial metric provided that for each $\alpha, \beta, \gamma \in \mathbb{V}$, the following conditions are satisfied:

($\mathtt{P}_1$) $\alpha = \beta \Leftrightarrow \mathtt{p}(\alpha,\alpha) = \mathtt{p}(\alpha,\beta) = \mathtt{p}(\beta,\beta)$;
($\mathtt{P}_2$) $\mathtt{p}(\alpha,\alpha) \leq \mathtt{p}(\alpha,\beta)$;
($\mathtt{P}_3$) $\mathtt{p}(\alpha,\beta) = \mathtt{p}(\beta,\alpha)$;
($\mathtt{P}_4$) $\mathtt{p}(\alpha,\beta) \leq \mathtt{p}(\alpha,\gamma) + \mathtt{p}(\gamma,\beta) - \mathtt{p}(\gamma,\gamma)$.

On this wise, $(\mathbb{V},\mathtt{p})$ is said to be a partial metric space (it is clear that if $\mathtt{p}(\alpha,\beta) = 0$, then from the axioms ($\mathtt{P}_1$) and ($\mathtt{P}_2$), $\alpha = \beta$) besides, every metric space is automatically a partial metric space, but the inverse does not hold.

Unless otherwise stated, partial metric space $(\mathbb{V},\mathtt{p})$ will be denoted by $\mathbb{V}_{\mathtt{p}}$ in the rest of the paper. Considering Definition 1, some examples of partial metric spaces are given below.

Let $\mathtt{p} : \mathbb{V}\times\mathbb{V} \to \mathbb{R}^{\geq 0}$ $\mathtt{p}(\alpha,\beta) = \max\{\alpha,\beta\}$ be a mapping such that $\mathtt{p}$ is partial metric on $\mathbb{R}^{\geq 0}$. However, $\mathtt{p}$ is not metric because ($\mathtt{P}_2$) property is not satisfied.

Let $\mathbb{V} = \{[c,d] : c \leq d\}$ with $c,d \in \mathbb{R}$ define $\mathtt{p} : \mathbb{V}\times\mathbb{V} \to \mathbb{R}^{\geq 0}$ such that $\mathtt{p}([c_1,c_2],[d_1,d_2]) = \max\{c_2,d_2\} - \min\{c_1,d_1\}$, then $(\mathbb{V},\mathtt{p})$ is a partial metric space.

Let $\mathtt{p}^w$ be a function on $\mathbb{V}\times\mathbb{V}$ such that

$$\mathtt{p}^w(\alpha,\beta) = 2\mathtt{p}(\alpha,\beta) - \mathtt{p}(\alpha,\alpha) - \mathtt{p}(\beta,\beta).$$

Then, $(\mathbb{V},\mathtt{p}^w)$ is a metric space.

With the function $\mathtt{p}^w$ defined above, a metric space is obtained from each partial metric space.

**Definition 2** ([17]). Let $\alpha \in \mathbb{V}$ and $\varepsilon > 0$. Then, the set

$$\mathcal{B}_p(\alpha,\varepsilon) = \{\beta \in \mathbb{V} : \mathtt{p}(\alpha,\beta) < \mathtt{p}(\alpha,\alpha) + \varepsilon\}$$

is called an open ball of radius $\varepsilon$ with center x.

Each partial metric $\mathtt{p}$ on $\mathbb{V}$ generates a $\tau_{\mathtt{p}}$ topology that takes as a base the family of $\mathtt{p}-$open balls on $\mathbb{V}$ for each element a $\tau_{\mathtt{p}}$ of $\mathbb{V}$ and $\varepsilon > 0$ with this $\tau_{\mathtt{p}}$ topology, $(\mathbb{V},\tau_{\mathtt{p}})$ is a $T_0$ space.

**Definition 3** ([17]). For a given $(\xi_k)$ sequence in $\mathbb{V}_{\mathtt{p}}$.

(1) $(\xi_k)$ is referred to as convergent to $\xi_0 \in \mathbb{V}$ if

$$\lim_{k\to\infty} \mathtt{p}(\xi_k,\xi_0) = \mathtt{p}(\xi_0,\xi_0)$$

or equivalently, for all $\varepsilon > 0$, there exists a $k_\varepsilon \in \mathbb{N}$ such that

$$|\mathtt{p}(\xi_k,\xi_0) - \mathtt{p}(\xi_0,\xi_0)| < \varepsilon$$

whenever $k > k_\varepsilon$.

(2) $(\xi_k)$ is properly convergent to $\xi_0 \in \mathbb{V}$, if $\lim_{k\to\infty} \mathtt{p}^w(\xi_k,\xi_0) = 0$.

(3) A sequence $(\xi_k)$ is called bounded in $\mathbb{V}_{\mathtt{p}}$ if, for all $k, l \in \mathbb{N}$, there exists a $R > 0$ such that $\mathtt{p}(\xi_k, \xi_l) < R$.

The statistical convergence in $\mathbb{V}_{\mathtt{p}}$ defined by Nuray [20] is given by definition (4).

**Definition 4** ([20]). A sequence $(\xi_k)$ in $\mathbb{V}_{\mathtt{p}}$ is said to be statistically convergent to $\xi_0 \in \mathbb{V}$ if, for all $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : |\mathtt{p}(\xi_k, \xi_0) - \mathtt{p}(\xi_0, \xi_0)| \geq \varepsilon\}) = 0.$$

The lacunary statistical convergence in $\mathbb{V}_{\mathtt{p}}$ conceptualized by Gülle et al. [?] is given by definition (5).

**Definition 5** ([?]). A sequence $(\xi_k)$ in $\mathbb{V}_{\mathtt{p}}$ is called lacunary statistically convergent to $\xi_0 \in \mathbb{V}$ if, for all $\varepsilon > 0$,

$$\lim_{r \to \infty} \frac{1}{h_r} |\{k \in I_r : |\mathtt{p}(\xi_k, \xi_0) - \mathtt{p}(\xi_0, \xi_0)| \geq \varepsilon\}| = 0.$$

It is convenient to represent symbolically by $S_\theta^{\mathtt{p}} - \lim_{k \to \infty} \mathtt{p}(\xi_k, \xi_0) = \mathtt{p}(\xi_0, \xi_0)$.

Çakı and Or [22] introduced asymptotically lacunary statistical equivalent sequences in partial metric space and examined the relations between them.

**Definition 6** ([22]). Let $\theta$ be a lacunary sequence and $\xi = (\xi_k)$, $\eta = (\eta_k)$ are non-negative sequences in $\mathbb{V}_{\mathtt{p}}$. These sequences are referred to as the asymptotically lacunary statistical equivalent of multiple $\alpha$ if, for all $\varepsilon > 0$ and $\lambda \in \mathbb{V}$,

$$\lim_{r \to \infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \mathtt{p}\left(\frac{\xi_k}{\eta_k}, \lambda\right) - \mathtt{p}(\lambda, \lambda) \right| \geq \varepsilon \right\} \right| = 0,$$

and is denoted by $\xi \overset{\mathtt{p}-S_\theta^\lambda}{\sim} \eta$. In addition, the set of aforesaid sequences is denoted by $\mathtt{p} - S_\theta^\lambda$. If $p(\lambda, \lambda) = 1$, the sequences mentioned are simply asymptotically lacunary statistical equivalent.

Kostyrko et al. [23] established ideal convergence by generalizing statistical convergence.

**Definition 7** ([23]). A family $\mathcal{I} \subset P(S)$, where $S \neq \emptyset$ and $P(S)$ is a power set of the set $\mathbb{X}$, has been called an ideal on $S$ provided, (i) $\emptyset \in \mathcal{I}$, (ii) $T_1, T_2 \in \mathcal{I} \Rightarrow T_1 \cup T_2 \in \mathcal{I}$, (iii) $T_1 \in \mathcal{I}, T_2 \subset T_1 \Rightarrow T_2 \in \mathcal{I}$.

If $\mathcal{I} \neq P(S)$, then $\mathcal{I}$ termed as a non-trivial ideal and further any non-trivial ideal $\mathcal{I}$ is referred to as an admissible ideal on $S$ if $\mathcal{I} \supset \{\{s\} : s \in S\}$

A family of a finite subset of $\mathbb{N}$, denoted by $\mathcal{I}_f$, is an admissible ideal in $\mathbb{N}$.

**Definition 8** ([23]). A nonempty class $\mathcal{F} \subset P(S)$, where $S \neq \emptyset$, has been referred to as filter on $S$ if, (i) $\emptyset \notin \mathcal{F}$, (ii) $T_1, T_2 \in \mathcal{F} \Rightarrow T_1 \cap T_2 \in \mathcal{F}$, (iii)

For each $T_1 \in \mathcal{F}, T_1 \subseteq T_2 \Rightarrow T_2 \in \mathcal{F}$. Every ideal $\mathcal{I}$ is associated with a filter $\mathcal{F}(\mathcal{I})$ which is given as follows:

$$\mathcal{F}(\mathcal{I}) = \{T \subset S : S \setminus T \in \mathcal{I}\}.$$

**Definition 9** ([21])**.** Any sequence $(\xi_k)$ from $\mathbb{V}_{\mathtt{p}}$ is known as ideal convergent ($\mathcal{I}$-convergent) to $\xi_0$ from $\mathbb{V}$ provided, for all $\varepsilon > 0$,

$$\{k \in \mathbb{N} : |\mathtt{p}(\xi_k, \xi_0) - \mathtt{p}(\xi_0, \xi_0)| \geq \varepsilon\} \in \mathcal{I}.$$

Here, $\xi_0$ is called the $\mathcal{I}$-limit of the sequence $(\xi_k)$.

Moreover, for details on ideal convergence, we refer [24–27] and many others.

## 2. Main results

In this section, we are introducing comprehensive definitions that generalize the asymptotically statistical equivalent and asymptotically lacunary equivalent notions for non-negative sequences in partial metric space $\mathbb{V}_{\mathtt{p}}$ with the help of ideals. Using the aforementioned definitions, we propound exhaustive main results of our study. Additionally, we deal with the relations between these notions.

**Definition 10.** A sequence $\xi = (\xi_k)$ is called $\mathcal{I}$-statistically convergent to $\xi_0$ provided, for all $\varepsilon > 0$ and $\delta > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \leq n : |\mathtt{p}(\xi_k, \xi_0) - \mathtt{p}(\xi_0, \xi_0)| \geq \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

It is convenient to represent symbolically by $\xi_k \xrightarrow{S(\mathcal{I})} \xi_0$. $S(\mathcal{I})$ will stand for the class of all $\mathcal{I}$-statistically convergent sequences.

**Definition 11.** For given a lacunary sequence $\theta = (k_r)$, the sequence $\xi = (\xi_k)$ in $\mathbb{V}_{\mathtt{p}}$ is called $\mathcal{I}$-lacunary statistically convergent to $\xi_0$ provided that, for all $\varepsilon > 0$ and $\delta > 0$, we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \left|\{k \in I_r : |\mathtt{p}(\xi_k, \xi_0) - \mathtt{p}(\xi_0, \xi_0)| \geq \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

It is convenient to represent symbolically by $\xi_k \xrightarrow{S_\theta(\mathcal{I})} \xi_0$. $S_\theta(\mathcal{I})$ will stand for the class of all $\mathcal{I}$-lacunary statistically convergent sequences.

**Definition 12.** For given a lacunary sequence $\theta = (k_r)$, the sequence $\xi = (\xi_k)$ in $\mathbb{V}_{\mathtt{p}}$ is referred to as strong $\mathcal{I}$-lacunary convergent to $\xi_0$ provided, for all $\varepsilon > 0$, we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |\mathtt{p}(\xi_k, \xi_0) - \mathtt{p}(\xi_0, \xi_0)| \geq \varepsilon\right\} \in \mathcal{I}.$$

It is convenient to represent symbolically by $\xi_k \xrightarrow{N_\theta(\mathcal{I})} \xi_0$. $N_\theta(\mathcal{I})$ will stand for the class of all strong $\mathcal{I}$-lacunary convergent sequences.

**Definition 13.** Two non-negative sequences $\xi=(\xi_k)$ and $\eta=(\eta_k)$ in $\mathbb{V}_{\mathtt{p}}$ are called $\mathcal{I}$-asymptotically statistical equivalent of multiple $\lambda$ if, for all $\varepsilon>0$, $\delta>0$ and $\lambda\in\mathbb{V}$, we have

$$\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\leq n:\left|\mathtt{p}\left(\frac{\xi_k}{\eta_k},\lambda\right)-\mathtt{p}\left(\lambda,\lambda\right)\right|\geq\varepsilon\right\}\right|\geq\delta\right\}\in\mathcal{I}.$$

This situation is denoted by $\xi\overset{\mathtt{p}-S^{\lambda}(\mathcal{I})}{\sim}\eta$ and simply $\mathcal{I}$-asymptotically statistical equivalent if $p(\lambda,\lambda)=1$.

It can be observed that when $\mathcal{I}=\mathcal{I}_{fin}$, the asymptotically statistical equivalent of multiple $\lambda$ and the $\mathcal{I}$-asymptotically statistical equivalent coincide.

**Definition 14.** Let $\xi=(\xi_k)$ and $\eta=(\eta_k)$ be two non-negative sequences in $\mathbb{V}_{\mathtt{p}}$. These sequences are called properly $\mathcal{I}$-asymptotically statistical equivalent of multiple $\lambda$ provided that, for all $\varepsilon>0$, $\delta>0$ and $\lambda\in\mathbb{V}$, we have

$$\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\leq n:\mathtt{p}^{w}\left(\frac{\xi_k}{\eta_k},\lambda\right)\geq\varepsilon\right\}\right|\geq\delta\right\}\in\mathcal{I}.$$

We abbreviate it as $\xi\overset{\mathtt{p}^{w}-S^{\lambda}(\mathcal{I})}{\sim}\eta$.

**Theorem 1.** *Let $\xi=(\xi_k)$ and $\eta=(\eta_k)$ be two non-negative sequences in $\mathbb{V}_{\mathtt{p}}$. Then,*

$$\xi\overset{\mathtt{p}^{w}-S^{\lambda}(\mathcal{I})}{\sim}\eta\Leftrightarrow\xi\overset{\mathtt{p}-S^{\lambda}(\mathcal{I})}{\sim}\eta.$$

*Proof.* Let $\xi\overset{\mathtt{p}^{w}-S^{\lambda}(\mathcal{I})}{\sim}\eta$. Then,

$$\left\{n\in N:\frac{1}{n}\left|\left\{k\leq n:\mathtt{p}^{w}\left(\frac{\xi_k}{\eta_k},\lambda\right)\geq\varepsilon\right\}\right|\geq\delta\right\}\in\mathcal{I}.$$

Let $Q=\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\leq n:\left|\mathtt{p}\left(\frac{\xi_k}{\eta_k},\lambda\right)-\mathtt{p}\left(\lambda,\lambda\right)\right|\geq\varepsilon\right\}\right|\geq\delta\right\}$. Using definition of $\mathbb{V}_{\mathtt{p}}$,

$$\begin{aligned}\varepsilon\leq\mathtt{p}\left(\frac{\xi_k}{\eta_k},\lambda\right)-\mathtt{p}(\lambda,\lambda)&=\mathtt{p}\left(\frac{\xi_k}{\eta_k},\lambda\right)-\mathtt{p}(\lambda,\lambda)+\mathtt{p}\left(\frac{\xi_k}{\eta_k},\frac{\xi_k}{\eta_k}\right)-\mathtt{p}\left(\frac{\xi_k}{\eta_k},\frac{\xi_k}{\eta_k}\right)\\&\leq 2\mathtt{p}\left(\frac{\xi_k}{\eta_k},\lambda\right)-\mathtt{p}\left(\frac{\xi_k}{\eta_k},\frac{\xi_k}{\eta_k}\right)-\mathtt{p}(\lambda,\lambda)\\&=\mathtt{p}^{w}\left(\frac{\xi_k}{\eta_k},\lambda\right),\end{aligned}$$

this from inequality

$$\begin{aligned}&\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\leq n:\left|\mathtt{p}\left(\frac{\xi_k}{\eta_k},\lambda\right)-\mathtt{p}\left(\lambda,\lambda\right)\right|\geq\varepsilon\right\}\right|\geq\delta\right\}\\&\subset\left\{n\in N:\frac{1}{n}\left|\left\{k\leq n:\mathtt{p}^{w}\left(\frac{\xi_k}{\eta_k},\lambda\right)\geq\varepsilon\right\}\right|\geq\delta\right\}\end{aligned}$$

and $Q \in \mathcal{I}$ is provided from the properties of the ideal and hypothesis. Consequently, $\xi \overset{\mathtt{p}-S^{\lambda}(\mathcal{I})}{\sim} \eta$. The converse of this theorem can be obtained similarly. □

**Definition 15.** For given a lacunary sequence $\theta$ and $\xi = (\xi_k)$, $\eta = (\eta_k)$ are non-negative sequences in $\mathbb{V}_{\mathtt{p}}$. These sequences are referred to as $\mathcal{I}$-asymptotically lacunary statistical equivalent of multiple $\lambda$ if we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : \left|\mathtt{p}\left(\frac{\xi_k}{\eta_k}, \lambda\right) - \mathtt{p}(\lambda, \lambda)\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}$$

holds, for all $\varepsilon > 0$, $\lambda \in \mathbb{V}$. We abbreviate it as $\xi \overset{\mathtt{p}-S^{\lambda}_{\theta}(\mathcal{I})}{\sim} \eta$. In addition, the set of aforesaid sequences is denoted by $\mathtt{p} - S^{\lambda}_{\theta}(\mathcal{I})$. If $\mathtt{p}(\lambda, \lambda) = 1$, then the mentioned sequences are called simply $\mathcal{I}$-asymptotically lacunary statistical equivalent.

**Definition 16.** For given a lacunary sequence $\theta$ and $\xi = (\xi_k)$, $\eta = (\eta_k)$ are non-negative sequences in $\mathbb{V}_{\mathtt{p}}$. These sequences are called properly $\mathcal{I}$-asymptotically lacunary statistical equivalent of multiple $\lambda$ if, for all $\varepsilon > 0$ and $\lambda \in \mathbb{V}$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : \mathtt{p}^{w}\left(\frac{\xi_k}{\eta_k}, \lambda\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

We abbreviate it as $\xi \overset{\mathtt{p}^{w}-S^{\lambda}_{\theta}(\mathcal{I})}{\sim} \eta$.

It can be observed that when $\mathcal{I} = \mathcal{I}_{fin}$, the asymptotically lacunary statistical equivalent of multiple $\lambda$ and the $\mathcal{I}$-asymptotically lacunary statistical equivalent coincide.

**Theorem 2.** *Let $\xi = (\xi_k)$ and $\eta = (\eta_k)$ be two non-negative sequences in $\mathbb{V}_{\mathtt{p}}$. Then,*

$$\xi \overset{\mathtt{p}^{w}-S^{\lambda}_{\theta}(\mathcal{I})}{\sim} \eta \Leftrightarrow \xi \overset{\mathtt{p}-S^{\lambda}_{\theta}(\mathcal{I})}{\sim} \eta.$$

*Proof.* This theorem can be proved similarly to the Theorem (1). □

**Definition 17.** For given a lacunary sequence $\theta$ and $\xi = (\xi_k)$, $\eta = (\eta_k)$ are non-negative sequences in $\mathbb{V}_{\mathtt{p}}$. These sequences are called strong $\mathcal{I}$-asymptotically lacunary equivalent of multiple $\lambda$ if, for all $\varepsilon > 0$ and $\lambda \in \mathbb{V}$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in I_r}\left|\mathtt{p}\left(\frac{\xi_k}{\eta_k}, \lambda\right) - p(\lambda, \lambda)\right| \geq \varepsilon\right\} \in \mathcal{I},$$

and is denoted by $\xi \overset{\mathtt{p}-N^{\lambda}_{\theta}(\mathcal{I})}{\sim} \eta$. In addition, the set of aforesaid sequences is denoted by $\mathtt{p} - N^{\lambda}_{\theta}(\mathcal{I})$. If $p(\lambda, \lambda) = 1$, then the mentioned sequences are called simply strong $\mathcal{I}$-asymptotically lacunary equivalent.

**Definition 18.** For given a lacunary sequence $\theta$ and $\xi = (\xi_k)$, $\eta = (\eta_k)$ are non-negative sequences in $\mathbb{V}_{\mathtt{p}}$. These sequences are called properly strong $\mathcal{I}$-asymptotically lacunary equivalent of multiple $\lambda$ if, for all $\varepsilon > 0$ and $\lambda \in \mathbb{V}$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathtt{p}^w \left( \frac{\xi_k}{\eta_k}, \lambda \right) = 0 \right\} \in \mathcal{I},$$

in $(\mathbb{V}, \mathtt{p}^w)$. We abbreviate it as $\xi \overset{\mathtt{p}^w - N_\theta^\lambda(\mathcal{I})}{\sim} \eta$.

In view of the mentioned definition, we obtain the next result.

**Theorem 3.** *Let $\xi = (\xi_k)$ and $\eta = (\eta_k)$ be two non-negative sequences in $\mathbb{V}_{\mathtt{p}}$. Then,*

$$\xi \overset{\mathtt{p}^w - N_\theta^\lambda(\mathcal{I})}{\sim} \eta \quad \Leftrightarrow \quad \xi \overset{\mathtt{p} - N_\theta^\lambda(\mathcal{I})}{\sim} \eta.$$

*Proof.* This theorem can be proved similarly to the theorem (1). □

**Theorem 4.** *For given a lacunary sequence $\theta = (k_r)$ and two non-negative sequences $\xi = (\xi_k)$ and $\eta = (\eta_k)$ in $\mathbb{V}_{\mathtt{p}}$. Then,*

(1) $\xi \overset{\mathtt{p} - N_\theta^\lambda(\mathcal{I})}{\sim} \eta \;\Rightarrow\; \xi \overset{\mathtt{p} - S_\theta^\lambda(\mathcal{I})}{\sim} \eta$,
(2) $\xi, \eta \in l_\infty$ *and* $\xi \overset{\mathtt{p} - S_\theta^\lambda(\mathcal{I})}{\sim} \eta \;\Rightarrow\; \xi \overset{\mathtt{p} - N_\theta^\lambda(\mathcal{I})}{\sim} \eta$,
(3) $S_\theta^\lambda(\mathcal{I}) \cap l_\infty \;=\; N_\theta^\lambda(\mathcal{I}) \cap l_\infty$.

*Proof.* (1) Assume that $\xi \overset{\mathtt{p} - N_\theta^\lambda(\mathcal{I})}{\sim} \eta$, for all $\varepsilon > 0$ and $\lambda \in \mathbb{V}$,

$$\sum_{k \in I_r} \left| \mathtt{p}\left( \frac{\xi_k}{\eta_k}, \lambda \right) - \mathtt{p}(\lambda, \lambda) \right| \geq \sum_{\substack{k \in I_r \\ \left| \mathtt{p}\left( \frac{\xi_k}{\eta_k}, \lambda \right) - \mathtt{p}(\lambda,\lambda) \right| \geq \varepsilon}} \left| \mathtt{p}\left( \frac{\xi_k}{\eta_k}, \lambda \right) - \mathtt{p}(\lambda, \lambda) \right|$$

$$\geq \varepsilon. \left| \left\{ k \in I_r : \left| \mathtt{p}\left( \frac{\xi_k}{\eta_k}, \lambda \right) - \mathtt{p}(\lambda, \lambda) \right| \geq \varepsilon \right\} \right|$$

and so

$$\frac{1}{\varepsilon h_r} \sum_{k \in I_r} \left| \mathtt{p}\left( \frac{\xi_k}{\eta_k}, \lambda \right) - \mathtt{p}(\lambda, \lambda) \right| \geq \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \mathtt{p}\left( \frac{\xi_k}{\eta_k}, \lambda \right) - \mathtt{p}(\lambda, \lambda) \right| \geq \varepsilon \right\} \right|.$$

Therefore, the following inclusion is provided for all $\delta > 0$.

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \mathtt{p}\left( \frac{\xi_k}{\eta_k}, \lambda \right) - \mathtt{p}\left( \lambda, \lambda \right) \right| \geq \varepsilon \right\} \right| \geq \delta \right\}$$

$$\subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| \mathtt{p}\left( \frac{\xi_k}{\eta_k}, \lambda \right) - p(\lambda, \lambda) \right| \geq \varepsilon.\delta \right\} \in \mathcal{I}.$$

Since $\xi \overset{\mathtt{p} - N_\theta^\lambda(\mathcal{I})}{\sim} \eta$, then $\xi \overset{\mathtt{p} - S_\theta^\lambda(\mathcal{I})}{\sim} \eta$.

(2) Let $\xi, \eta \in l_\infty$ and $\xi \overset{\mathtt{p}-S_\theta^\lambda(\mathcal{I})}{\sim} \eta$. Then, for all $\varepsilon > 0$, $\delta > 0$ and $\lambda \in \mathbb{V}$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : \left|\mathtt{p}\left(\frac{\xi_k}{\eta_k}, \lambda\right) - \mathtt{p}\left(\lambda, \lambda\right)\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}$$

and there exists $C > 0$ such that

$$\left|\mathtt{p}\left(\frac{\xi_k}{\eta_k}, \lambda\right) - \mathtt{p}(\lambda, \lambda)\right| \leq C.$$

For all $\varepsilon > 0$,

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k \in I_r}\left|\mathtt{p}\left(\frac{\xi_k}{\eta_k}, \lambda\right) - \mathtt{p}(\lambda, \lambda)\right| \\
&= \frac{1}{h_r}\sum_{\substack{k \in I_r \\ \left|\mathtt{p}\left(\frac{\xi_k}{\eta_k}, \lambda\right) - \mathtt{p}(\lambda,\lambda)\right| \geq \varepsilon}}\left|\mathtt{p}\left(\frac{\xi_k}{\eta_k}, \lambda\right) - \mathtt{p}(\lambda, \lambda)\right| \\
&+ \frac{1}{h_r}\sum_{\substack{k \in I_r \\ \left|\mathtt{p}\left(\frac{\xi_k}{\eta_k}, \lambda\right) - \mathtt{p}(\lambda,\lambda)\right| < \varepsilon}}\left|\mathtt{p}\left(\frac{\xi_k}{\eta_k}, \lambda\right) - \mathtt{p}(\lambda, \lambda)\right| \\
&\leq \frac{M}{h_r}\left|\left\{k \in I_r : \left|\mathtt{p}\left(\frac{\xi_k}{\eta_k}, \lambda\right) - \mathtt{p}(\lambda, \lambda)\right| \geq \frac{\varepsilon}{2}\right\}\right| + \frac{\varepsilon}{2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in I_r}\left|\mathtt{p}\left(\frac{\xi_k}{\eta_k}, \lambda\right) - p(\lambda, \lambda)\right| \geq \varepsilon\right\} \\
&\subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : \left|\mathtt{p}\left(\frac{\xi_k}{\eta_k}, \lambda\right) - \mathtt{p}\left(\lambda, \lambda\right)\right| \geq \frac{\varepsilon}{2}\right\}\right| \geq \frac{\varepsilon}{2C}\right\} \in \mathcal{I}.
\end{aligned}$$

Consequently, $\xi \overset{\mathtt{p}-N_\theta^\lambda(\mathcal{I})}{\sim} \eta$.

(3) It is clear from (1) and (2). □

**Theorem 5.** *Let $\xi = (\xi_k)$, $\eta = (\eta_k)$ are two non-negative sequences in $\mathbb{V}_{\mathtt{p}}$, $\mathcal{I}$ is an ideal and $\theta$ be a lacunary sequence with $\liminf q_r > 1$. If $\xi \overset{\mathtt{p}-S^\lambda(\mathcal{I})}{\sim} \eta$, then $\xi \overset{\mathtt{p}-S_\theta^\lambda(\mathcal{I})}{\sim} \eta$.*

*Proof.* Let $\liminf q_r > 1$. Then, there exists a $\ell > 0$ such that $q_r \geq 1 + \ell$ for sufficiently large r, which signify $\frac{h_r}{k_r} \geq \frac{\ell}{1+\ell}$. If $\xi \overset{\mathtt{p}-S_\theta^\lambda(\mathcal{I})}{\sim} \eta$, then for all $\varepsilon > 0$ and $\lambda \in \mathbb{V}$, and for sufficiently large r, we have

$$\begin{aligned}
&\frac{1}{k_r}\left|\left\{k \leq k_r : \left|\mathtt{p}\left(\frac{\xi_k}{\eta_k}, \lambda\right) - \mathtt{p}(\lambda, \lambda)\right| \geq \varepsilon\right\}\right| \\
&\geq \frac{1}{k_r}\left|\left\{k \in I_r : \left|\mathtt{p}\left(\frac{\xi_k}{\eta_k}, \lambda\right) - \mathtt{p}(\lambda, \lambda)\right| \geq \varepsilon\right\}\right|
\end{aligned}$$

$$\geq \frac{\ell}{1+\ell}\frac{1}{h_r}\left|\left\{k \in I_r : \left|\mathtt{p}\left(\frac{\xi_k}{\eta_k},\lambda\right) - \mathtt{p}(\lambda,\lambda)\right| \geq \varepsilon\right\}\right|.$$

Then, for all $\delta > 0$, we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : \left|\mathtt{p}\left(\frac{\xi_k}{\eta_k},\lambda\right) - \mathtt{p}\left(\lambda,\lambda\right)\right| \geq \varepsilon\right\}\right| \geq \delta\right\}$$

$$\subseteq \left\{r \in \mathbb{N} : \frac{1}{k_r}\left|\left\{k \in I_r : \left|\mathtt{p}\left(\frac{\xi_k}{\eta_k},\lambda\right) - \mathtt{p}\left(\lambda,\lambda\right)\right| \geq \varepsilon\right\}\right| \geq \frac{\delta\ell}{1+\ell}\right\} \in \mathcal{I}.$$

Since $\xi \overset{\mathtt{p}-S^{\lambda}(\mathcal{I})}{\sim} \eta$, then $(\xi_k)$ and $(\eta_k)$ are $\mathcal{I}$-asymptotically lacunary statistical equivalent of multiple $\lambda$. □

To state Theorem (6), we granted that the lacunary sequence $\theta$ fulfills the following condition for any set $K \in \mathcal{F}(\mathcal{I})$.

$$\bigcup\{n : k_{r-1} < n < k_r, r \in K\} \in \mathcal{F}(\mathcal{I}).$$

**Theorem 6.** *Let $\xi = (\xi_k)$, $\eta = (\eta_k)$ are two non-negative sequences in $\mathbb{V}_{\mathtt{p}}$, $\mathcal{I}$ is an ideal and $\theta$ be a lacunary sequence with* $\limsup q_r < \infty$. *If $\xi \overset{\mathtt{p}-S_{\theta}^{\lambda}(\mathcal{I})}{\sim} \eta$, then $\xi \overset{\mathtt{p}-S^{\lambda}(\mathcal{I})}{\sim} \eta$.*

*Proof.* Let $\limsup q_r < \infty$, then there exists $M > 0$ such that $q_r < M$, for all $r \geq 1$. Let $\xi \overset{\mathtt{p}-S_{\theta}^{\lambda}(\mathcal{I})}{\sim} \eta$ and $\varepsilon, \delta, \delta_1 > 0$ and $\lambda \in \mathbb{V}$ define the sets

$$K = \left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : \left|\mathtt{p}\left(\frac{\xi_k}{\eta_k},\lambda\right) - \mathtt{p}\left(\lambda,\lambda\right)\right| \geq \varepsilon\right\}\right| < \delta\right\}$$

and

$$S = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \left|\mathtt{p}\left(\frac{\xi_k}{\eta_k},\lambda\right) - \mathtt{p}\left(\lambda,\lambda\right)\right| \geq \varepsilon\right\}\right| < \delta_1\right\}.$$

Our presumption that $K \in \mathcal{F}(\mathcal{I})$, the filter connected to the ideal $\mathcal{I}$, is clear. Additionally, note that

$$\gamma_j = \frac{1}{h_j}\left|\left\{k \in I_j : \left|\mathtt{p}\left(\frac{\xi_k}{\eta_k},\lambda\right) - \mathtt{p}\left(\lambda,\lambda\right)\right| \geq \varepsilon\right\}\right| < \delta$$

for all $j \in K$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n < k_r$ for some $r \in K$. Now

$$\begin{aligned}
&\frac{1}{n}\left|\left\{k \leq n : \left|\mathtt{p}\left(\frac{\xi_k}{\eta_k},\lambda\right) - \mathtt{p}(\lambda,\lambda)\right| \geq \varepsilon\right\}\right| \\
&\leq \frac{1}{k_{r-1}}\left|\left\{k \leq k_r : \left|\mathtt{p}\left(\frac{\xi_k}{\eta_k},\lambda\right) - \mathtt{p}(\lambda,\lambda)\right| \geq \varepsilon\right\}\right| \\
&= \frac{1}{k_{r-1}}\left|\left\{k \in I_1 : \left|\mathtt{p}\left(\frac{\xi_k}{\eta_k},\lambda\right) - \mathtt{p}(\lambda,\lambda)\right| \geq \varepsilon\right\}\right| \\
&\quad + \frac{1}{k_{r-1}}\left|\left\{k \in I_2 : \left|\mathtt{p}\left(\frac{\xi_k}{\eta_k},\lambda\right) - \mathtt{p}(\lambda,\lambda)\right| \geq \varepsilon\right\}\right|
\end{aligned}$$

$$
\begin{aligned}
&+\cdots+\frac{1}{k_{r-1}}\left|\left\{k\in I_r:\left|\mathtt{p}\left(\frac{\xi_k}{\eta_k},\lambda\right)-\mathtt{p}(\lambda,\lambda)\right|\geq\varepsilon\right\}\right| \\
&=\frac{k_1}{k_{r-1}}\frac{1}{h_1}\left|\left\{k\in I_1:\left|\mathtt{p}\left(\frac{\xi_k}{\eta_k},\lambda\right)-\mathtt{p}(\lambda,\lambda)\right|\geq\varepsilon\right\}\right| \\
&+\frac{k_2-k_1}{k_{r-1}}\frac{1}{h_2}\left|\left\{k\in I_2:\left|\mathtt{p}\left(\frac{\xi_k}{\eta_k},\lambda\right)-\mathtt{p}(\lambda,\lambda)\right|\geq\varepsilon\right\}\right| \\
&+\cdots+\frac{k_r-k_{r-1}}{k_{r-1}}\frac{1}{h_r}\left|\left\{k\in I_r:\left|\mathtt{p}\left(\frac{\xi_k}{\eta_k},\lambda\right)-\mathtt{p}(\lambda,\lambda)\right|\geq\varepsilon\right\}\right| \\
&=\frac{k_1}{k_{r-1}}\gamma_1+\frac{k_2-k_1}{k_{r-1}}\gamma_2+\cdots+\frac{k_r-k_{r-1}}{k_{r-1}}\gamma_r \\
&\leq\sup_{j\in K}\gamma_j\frac{k_r}{k_{r-1}}<M\delta.
\end{aligned}
$$

Choosing $\delta_1=\frac{\delta}{M}$ and taking into account that $\bigcup\{n:k_{r-1}<n<k_r,r\in K\}\subset S$, it follows from our lacunary sequence $\theta$ hypothesis that $\mathcal{F}(\mathcal{I})$ also includes the set $S$. Accordingly, $\xi\overset{\mathtt{p}-S^{\lambda}(\mathcal{I})}{\sim}\eta$. □

**Corollary 1.** *Let* $\xi=(\xi_k)$, $\eta=(\eta_k)$ *are two non-negative sequences in* $\mathbb{V}_{\mathtt{p}}$, $\mathcal{I}$ *is an ideal and* $\theta$ *be a lacunary sequence with* $1<\liminf q_r<\limsup q_r<\infty$. *Then,* $\xi\overset{\mathtt{p}-S^{\lambda}_{\theta}(\mathcal{I})}{\sim}\eta=\xi\overset{\mathtt{p}-S^{\lambda}(\mathcal{I})}{\sim}\eta$.

## 3. Conclusion

The present study contributes to the broader studies of the notion of summability in partial metric spaces. In the mentioned spaces, we introduce the concepts of $\mathcal{I}$-asymptotically statistical equivalent and $\mathcal{I}$-asymptotically lacunary statistical equivalent using asymptotically equivalent and lacunary sequence. By examining some properties related to the lacunary statistical equivalent in these spaces, we have established equivalent conditions on $\mathcal{I}$-Lacunary statistical equivalent in spaces aforesaid for ideal statistically convergent sequences. Afterward, we investigate their relationship and make some observations related to asymptotically equivalent types using the tools of partial metric space. In future work, it would be a valuable motivation for the authors to study the properties of the types of convergence we have defined in partial metric space. Moreover, in light of the mentioned studies, the same concepts can be extended to sequences of sets.

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