Mathematica Moravica
Vol. **28**, No. **2** (2024), 85–102 doi: 10.5937/MatMor2402085K

# Clairaut slant submersion from almost Hermitian manifolds

Sushil Kumar, Rajendra Prasad, Punit Kumar Singh

Abstract. Our main aim is to introduce Clairaut slant submersions in complex geometry. We give the notion of Clairaut slant submersions from almost Hermitian manifolds onto Riemannian manifold in this article. We obtain some basic results on discussed submersions. Furthermore, we provide some examples to explore the geometry of Clairaut slant submersions.

## 1. Introduction

Let $M$ be a Riemannian manifold endowed with a Riemannian metric $g$. An almost Hermitian manifold is a subclass of almost complex manifold. Since the Riemannian submersions have many applications in science and technology, especially in the theory of relativity and cosmology, many researchers are attracted to this area.

In 1966 the theory of Riemannian submersion was initiated by O' Neill [15] and it has been further studied by Gray [8], in 1967. Later, Watson [30] defined almost Hermition submersions and showed that horizontal and vertical distributions are invariant with respect to the almost complex structure. The Riemannian submersions play a vital role not only in the differential geometry but also in science and technology. It is noticed that the theory of Riemannian submersions are capable of handling many issues of the singularity theory, Yang-Mills theory, quantum theory, Kaluza-Klein theory, relativity, superstring theories, etc. (see, [2, 6, 9]). For more details, we cite the books ([7, 23]) and the references therein. The Riemannian submersions motivate the researchers to define the anti-invariant submersion [24], semi-invariant submersion [26], invariant submersions [23], slant submersions [25], semi-slant submersions [16], conformal anti-invariant submersions ([11, 17]), conformal semi-slant submersions [20], quasi-bi-slant submersions [18], (for further details, see [10, 19, 21, 22]), etc.

2020 *Mathematics Subject Classification.* Primary: 53C12; Secondary: 53C55.
*Key words and phrases.* Kähler manifold, Riemannian submersions, Clairaut slant submersions.
*Full paper.* Received 22 June 2024, accepted 7 November 2024, available online 4 December 2024.

In 1972 Bishop [4] introduced and studied a new and interesting class of Riemannian submersion as: if there is a function $r : M \to R^+$ such that for every geodesic, making an angle $\theta$ with the horizontal subspaces, $rsin\theta$ is constant, then submersion $\pi : M \to N$ is said to be a Clairaut submersion. Afterwards, this notion has been widely studied in Lorentzian spaces [1], timelike and spacelike spaces [14], static spacetimes ([28, 29]). In 1991 Aso et al. [3] generalized Clairaut submersions and the new conditions for anti-invariant Riemannian submersions to be Clairaut were described in [14]. In 2017 Sahin introduced Clairaut Riemannian map [27] and studied its geometric properties and S. Kumar et al. [13] studied pointwise slant submersions from Kenmotsu manifolds. Recently, Yadav and Meena [31] have defined Clairaut anti-invariant Riemannian maps from Kahler manifolds and Kumar et al. studied Clairaut semi-invariant Riemannian maps in [12].

The above studies inspire us to introduce the notion of Clairaut slant submersions from Hermitian manifolds onto Riemannian manifolds. We exhibit our work as follows: section 2, contains some basic concepts which are needed further in the paper. In section 3, we define the Clairaut slant submersions from Kähler manifolds onto Riemannian manifolds and discuss the differential geometric properties of such submersions. Curvature Relations of Clairaut slant submersions are discussed in section 4 and the last section contains some explicit examples of discussed submersions.

## 2. PRELIMINARIES

Let $J$ be a $(1, 1)$ tensor field on an even-dimensional differentiable manifold $N_1$ and $I$ is identity operator in such a manner that

$$J^2 = -I. \tag{1}$$

Then $J$ is called an almost complex structure on $N_1$. The manifold $N_1$ with an almost complex structure $J$ is called an almost complex manifold. Nijenhuis tensor $N$ of an almost complex structure is defined as:

$$N(V_1, W_1) = [JV_1, JW_1] - [V_1, W_1] - J[JV_1, W_1] - J[V_1, JW_1], \tag{2}$$

for all $V_1, W_1 \in \Gamma(TN_1)$.

The almost complex manifold $N_1$ is called a complex manifold, if $N$ vanishes on an almost complex manifold $N_1$.

Let $g_1$ be a Riemannian metric on $N_1$, then $g_1$ is called a Hermitian metric on $N_1$ if

$$g_1(JZ_1, JW_1) = g_1(Z_1, W_1), \quad \text{for all } Z_1, W_1 \in \Gamma(TN_1). \tag{3}$$

Now, manifold $N_1$ with Hermitian metric $g_1$ is called an almost Hermitian manifold. The Riemannian connection $\nabla$ of the $N_1$ can be extended to the whole tensor algebra on $N_1$. Tensor fields $(\nabla_{Y_1} J)$ is defined as

$$(\nabla_{Y_1} J)Z_1 = \nabla_{Y_1} JZ_1 - J\nabla_{Y_1} Z_1, \tag{4}$$

for all $Y_1, Z_1 \in \Gamma(TN_1)$.

An almost Hermitian manifold $(N_1, g_1, J)$ is called a Kähler manifold [5] if

$$(\nabla_{Y_1} J)Z_1 = 0, \tag{5}$$

for all $Y_1, Z_1 \in \Gamma(TN_1)$.

For a Kähler manifold $(N_1, g_1, J)$ we have

$$R(X_1, X_2, X_3, X_4) = R(JX_1, JX_2, JX_3, JX_4), \tag{6}$$

$$R(X_1, X_2, X_3, X_4) = R(X_1, X_2, JX_3, JX_4), \tag{7}$$

$$R(X_1, X_2, JX_3, X_4) = -R(X_1, X_2, X_3, JX_4), \tag{8}$$

$$R(JX_1, X_2, JX_3, X_4) = R(X_1, JX_2, X_3, JX_4), \tag{9}$$

$$\begin{aligned} R(X_1, X_2, X_3, X_4) = R(JX_1, JX_2, X_3, X_4) + R(JX_1, X_2, JX_3, X_4) \\ + R(JX_1, X_2, X_3, JX_4) \end{aligned} \tag{10}$$

for all $X_1, X_2, X_3, X_4 \in \Gamma(TN_1)$, where $R(X_1, X_2)X_3 = \nabla_{X_1}\nabla_{X_2}X_3 - \nabla_{X_2}\nabla_{X_1}X_3 - \nabla_{[X_1,X_2]}X_3$ denotes the Riemannian curvature tensor filed of $N_1$.

Define O'Neill's tensors [15] $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_{E_1}E_2 = \mathcal{H}\nabla_{\mathcal{H}E_1}\mathcal{V}E_2 + \mathcal{V}\nabla_{\mathcal{H}E_1}\mathcal{H}E_2, \tag{11}$$

$$\mathcal{T}_{E_1}E_2 = \mathcal{H}\nabla_{\mathcal{V}E_1}\mathcal{V}E_2 + \mathcal{V}\nabla_{\mathcal{V}E_1}\mathcal{H}E_2, \tag{12}$$

for any vector fields $E_1, E_2$ on $N_1$, where $\nabla$ is the Levi-Civita connection of $g_1$. It is easy to see that $\mathcal{T}_{E_1}$ and $\mathcal{A}_{E_1}$ are skew-symmetric operators on the tangent bundle of $N_1$ reversing the vertical and the horizontal distributions.

From equations (11) and (12), we have

$$\nabla_{Z_1}U_1 = \mathcal{T}_{Z_1}U_1 + \mathcal{V}\nabla_{Z_1}U_1, \tag{13}$$

$$\nabla_{Z_1}W_1 = \mathcal{T}_{Z_1}W_1 + \mathcal{H}\nabla_{Z_1}W_1, \tag{14}$$

$$\nabla_{W_1}Z_1 = \mathcal{A}_{W_1}Z_1 + \mathcal{V}\nabla_{W_1}Z_1, \tag{15}$$

$$\nabla_{V_1}W_1 = \mathcal{H}\nabla_{V_1}W_1 + \mathcal{A}_{V_1}W_1, \tag{16}$$

for all $Z_1, U_1 \in \Gamma(\ker F_*)$ and $V_1, W_1 \in \Gamma(\ker F_*)^\perp$, where $\mathcal{H}\nabla_{Z_1}W_1 = \mathcal{A}_{W_1}Z_1$, if $W_1$ is basic. It is clear that $\mathcal{T}$ performs on the fibers as the second fundamental form, while $\mathcal{A}$ performs on the horizontal distribution and measures the obstruction to the integrability of this distribution.

A differentiable map $F$ between two Riemannian manifolds is totally geodesic if

$$(\nabla F_*)(Z_1, Z_2) = 0, \quad \text{for all } Z_1, Z_2 \in \Gamma(TN_1).$$

A totally geodesic map is the one which maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

A Riemannian submersion is called a Riemannian submersion with totally umbilical fibers if [26]

$$\mathcal{T}_{X_1}Y_1 = g_1(X_1, Y_1)H, \tag{17}$$

for all $X_1, Y_1 \in \Gamma(\ker \digamma_*)$, where $H$ is the mean curvature vector field of fibers.

Let $\digamma : (N_1, g_1) \to (N_2, g_2)$ is a smooth map. Then $\digamma_*$ of $\digamma$ can be observed as a section of the bundle $Hom(TN_1, \digamma^{-1}TN_2) \to N_1$, where $\digamma^{-1}TN_2$ is the bundle which has fibers $(\digamma^{-1}TN_2)_x = T_{\digamma(x)}N_2$ has a connection $\nabla$ induced from the Riemannian connection $\nabla^{N_1}$ and the pullback connection. Then the second fundamental form of $\digamma$ is given by

$$(\nabla \digamma_*)(Z_1, V_1) = \nabla^{\digamma}_{Z_1}\digamma_*(V_1) - \digamma_*(\nabla^{N_1}_{Z_1}V_1), \tag{18}$$

for vector field $Z_1, V_1 \in \Gamma(TN_1)$, where $\nabla^{\digamma}$ is the pullback connection. We know that the second fundamental form is symmetric.

Now, we recall following definitions for further use:

**Definition 1** ([23])**.** Let $\digamma$ be a Riemannian submersion from an almost Hermitian manifold $(N_1, g_1, J)$ onto a Riemannian manifold $(N_2, g_2)$. Then, we say that $\digamma$ is an invariant submersion if the vertical distribution is invariant with respect to the complex structure $J$, i.e.,

$$J(\ker \digamma_*) = \ker \digamma_*.$$

**Definition 2** ([24])**.** Let $N_1$ be an almost Hermitian manifold with Hermitian metric $g_1$ and almost complex structure $J$ and $N_2$ be a Riemannian manifold with Riemannian metric $g_2$. Suppose that there exists a submersion $\digamma : (N_1, g_1, J) \to (N_2, g_2)$ such that $J(\ker \digamma_*) \subseteq (\ker \digamma_*)^{\perp}$. Then we say that $\digamma$ is an anti-invariant submersion.

**Definition 3** ([25])**.** Let $\digamma$ be a Riemannian map from an almost Hermitian manifold $(N_1, g_1, J)$ to a Riemannian manifold $(N_2, g_2)$. If for any non-zero vector $Z \in (\ker \digamma_*)$, the angle $\Theta(Z)$ between $JZ$ and the space $(\ker \digamma_*)$ is a constant, i.e., it is independent of the choice of the point $p \in N_1$ and choice of the tangent vector $Z$ in $(\ker \digamma_*)$, then we say that $\digamma$ is a slant submersion. In this case, the angle $\Theta$ is called the slant angle of the slant submersion.

## 3. CLAIRAUT SLANT SUBMERSIONS

Bishop [4] gave the notion of Clairaut Riemannian submersion. He defined that a Riemannian submersion $\digamma : (N_1, g_1) \to (N_2, g_2)$ is called a Clairaut Riemannian submersion if there exists a positive function $r$ on $N_1$, such that for any geodesic $\alpha$ on $N_1$, the function $(r \circ \alpha)\sin\theta$ is constant, where for any $t$, $\theta(t)$ is the angle between $\dot{\alpha}(t)$ and the horizontal space at $\alpha(t)$.

The necessary and sufficient condition for a Riemannian submersion to be a Clairaut Riemannian submersion was also given by Bishop as follows.

**Theorem 1** ([4]). *Let $\digamma : (N_1, g_1) \to (N_2, g_2)$ be a submersion with connected fibers. Then, $\digamma$ is a Clairaut submersion with $r = e^h$ if each fiber is totally umbilical and has the mean curvature vector field $H = -\nabla h$ with respect to $g_1$.*

Now, we present the notion of Clairaut slant submersion as follows.

**Definition 4.** Let $(N_1, g_1, J)$ be a Kähler manifold and $(N_2, g_2)$ be a Riemannian manifold. Any slant submersion from $(N_1, g_1, J)$ onto $(N_2, g_2)$ is called Clairaut slant submersion if it satisfies the condition of Clairaut submersion.

We denote the complementary distribution to $\omega(\ker \digamma_*)$ in $(\ker \digamma_*)^\perp$ by $\mu$. Then for $X_1 \in (\ker \digamma_*)$, we get

$$JX_1 = \phi X_1 + \omega X_1, \tag{19}$$

where $\phi X_1$ and $\omega X_1$ are vertical and horizontal parts of $JX_1$. Also for $X_2 \in \Gamma(\ker \digamma_*)^\perp$, we have

$$JX_2 = BX_2 + CX_2, \tag{20}$$

where $BX_2$ and $CX_2$ are vertical and horizontal components of $JX_2$.

The proof of the following result is the same as given in [25], therefore, we omit its proof.

**Lemma 1.** *Let $\digamma$ be a slant submersion from an almost Hermitian manifold $(N_1, g_1, J)$ onto a Riemannian manifold $(N_2, g_2)$. Then, we have*

(i) $\phi_1^2 W_1 = -(\cos^2 \Theta_1) W_1$,
(ii) $g_1(\phi W_1, \phi W_2) = \cos^2 \Theta_1 g_1(W_1, W_2)$,
(iii) $g_1(\omega W_1, \omega W_2) = \sin^2 \Theta_1 g_1(W_1, W_2)$,

*for all $W_1, W_2 \in \Gamma(\ker \digamma_*)$.*

**Lemma 2.** *Let $\digamma$ be a slant submersion from a Kähler manifold $(N_1, g_1, J)$ onto a Riemannian manifold $(N_2, g_2)$. Then, we have*

$$\mathcal{V}\nabla_{Y_1}\phi Y_2 + \mathcal{T}_{Y_1}\omega Y_2 = B\mathcal{T}_{Y_1}Y_2 + \phi\mathcal{V}\nabla_{Y_1}Y_2, \tag{21}$$

$$\mathcal{T}_{Y_1}\phi Y_2 + \mathcal{H}\nabla_{Y_1}\omega Y_2 = C\mathcal{T}_{Y_1}Y_2 + \omega\mathcal{V}\nabla_{Y_1}Y_2, \tag{22}$$

$$\mathcal{V}\nabla_{Y_1}BW_1 + \mathcal{T}_{Y_1}CW_1 = \phi\mathcal{T}_{Y_1}W_1 + B\mathcal{H}\nabla_{Y_1}W_1, \tag{23}$$

$$\mathcal{T}_{Y_1}BW_1 + \mathcal{H}\nabla_{Y_1}CW_1 = \omega\mathcal{T}_{Y_1}W_1 + C\mathcal{H}\nabla_{Y_1}W_1, \tag{24}$$

$$\mathcal{V}\nabla_{W_1}\phi Y_1 + \mathcal{A}_{W_1}\omega Y_1 = B\mathcal{A}_{W_1}Y_1 + \phi\mathcal{V}\nabla_{W_1}Y_1, \tag{25}$$

$$\mathcal{A}_{W_1}\phi Y_1 + \mathcal{H}\nabla_{W_1}\omega Y_1 = \omega\mathcal{V}_{W_1}Y_1 + C\mathcal{A}_{W_1}Y_1, \tag{26}$$

$$\mathcal{V}\nabla_{W_1}BW_2 + \mathcal{A}_{W_1}CW_2 = B\mathcal{H}\nabla_{W_1}W_2 + \phi\mathcal{A}_{W_1}W_2, \tag{27}$$

$$\mathcal{A}_{W_1}BW_2 + \mathcal{H}\nabla_{W_1}CW_2 = \omega\mathcal{A}_{W_1}W_2 + C\mathcal{H}\nabla_{W_1}W_2, \tag{28}$$

*for any $Y_1, Y_2 \in \Gamma(\ker \digamma_*)$ and $W_1, W_2 \in \Gamma(\ker \digamma_*)^\perp$.*

*Proof.* Using equations (13)-(16), (19) and (20), we get the lemma completely. □

Now, we define

$$(\nabla_{Y_1}\phi)Y_2 = \mathcal{V}\nabla_{Y_1}\phi Y_2 - \phi\mathcal{V}\nabla_{Y_1}Y_2, \tag{29}$$

$$(\nabla_{Y_1}\omega)Y_2 = \mathcal{H}\nabla_{Y_1}\omega Y_2 - \omega\mathcal{V}\nabla_{Y_1}Y_2, \tag{30}$$

$$(\nabla_{W_1}C)W_2 = \mathcal{H}\nabla_{W_1}CW_2 - C\mathcal{H}\nabla_{W_1}W_2, \tag{31}$$

$$(\nabla_{W_1}B)W_2 = \mathcal{V}\nabla_{W_1}BW_2 - B\mathcal{H}\nabla_{W_1}W_2, \tag{32}$$

for any $Y_1, Y_1 \in \Gamma(\ker F_*)$ and $W_1, W_2 \in \Gamma(\ker F_*)^{\perp}$.

**Lemma 3.** *Let $F$ be a slant submersion from a Kähler manifold $(N_1, g_1, J)$ onto a Riemannian manifold $(N_2, g_2)$. Then, we have*

$$\begin{aligned}(\nabla_{Y_1}\phi)Y_2 &= B\mathcal{T}_{Y_1}Y_2 - \mathcal{T}_{Y_1}\omega Y_2,\\ (\nabla_{Y_1}\omega)Y_2 &= C\mathcal{T}_{Y_1}Y_2 - \mathcal{T}_{Y_1}\phi Y_2,\\ (\nabla_{W_1}C)W_2 &= \omega\mathcal{A}_{W_1}W_2 - \mathcal{A}_{W_1}BW_2,\\ (\nabla_{W_1}B)W_2 &= \phi\mathcal{A}_{W_1}W_2 - \mathcal{A}_{W_1}CW_2,\end{aligned}$$

*for any vectors $Y_1, Y_2 \in \Gamma(\ker F_*)$ and $W_1, W_2 \in \Gamma(\ker F_*)^{\perp}$.*

*Proof.* The proof of the above lemma is straightforward, so we omit its proof. □

If the tensors $\phi$ and $\omega$ are parallel with respect to the linear connection $\nabla$ on $N_1$ respectively, then

$$B\mathcal{T}_{Y_1}Y_2 = \mathcal{T}_{Y_1}\omega Y_2, C\mathcal{T}_{Y_1}Y_2 = \mathcal{T}_{Y_1}\phi Y_2$$

for any $Y_1, Y_2 \in \Gamma(TN_1)$.

**Lemma 4.** *Let $F$ be a slant submersion from a Kähler manifold $(N_1, g_1, J)$ onto a Riemannian manifold $(N_2, g_2)$. If $\alpha : I \subset R \to M$ is a regular curve and $Y_1(t)$ and $Y_2(t)$ are the vertical and horizontal components of the tangent vector field $\dot{\alpha} = E$ of $\alpha(t)$, respectively, then $\alpha$ is a geodesic if and only if along $\alpha$ the following equations hold:*

$$\begin{aligned}\cos^2\Theta\mathcal{V}\nabla_{\dot{\alpha}}Y_1 = {} & \mathcal{T}_{Y_1}\omega\phi Y_1 + \mathcal{A}_{Y_2}\omega\phi Y_1 + \phi\mathcal{T}_{Y_1}\omega Y_1 + B\mathcal{H}\nabla_{Y_1}\omega Y_1\\ & + B\mathcal{T}_{Y_1}BY_2 + \phi\mathcal{V}\nabla_{Y_2}\omega Y_1 + \phi\mathcal{T}_{Y_1}CY_2 + \omega\mathcal{T}_{Y_1}CY_2\\ & + B\mathcal{H}\nabla_{Y_1}CY_2 + \phi\mathcal{A}_{Y_2}\omega Y_1 + B\mathcal{A}_{Y_2}BY_2 + \phi\mathcal{V}\nabla_{Y_2}BY_2\\ & + B\mathcal{H}\nabla_{Y_2}CY_2 + \phi\mathcal{A}_{Y_2}CY_2,\end{aligned}$$

$$\begin{aligned}\cos^2\Theta(\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})Y_1 = {} & \mathcal{H}\nabla_{Y_1}\omega\phi Y_1 + \mathcal{H}\nabla_{Y_2}\omega\phi Y_1 + \omega\mathcal{T}_{Y_1}\omega Y_1 + C\mathcal{H}\nabla_{Y_1}\omega Y_1\\ & + C\mathcal{T}_{Y_1}BY_2 + \omega\mathcal{V}\nabla_{Y_2}\omega Y_1 + \omega\mathcal{T}_{Y_1}CY_2 + C\mathcal{H}\nabla_{Y_1}CY_2\\ & + C\mathcal{H}\nabla_{Y_2}\omega Y_1 + \omega\mathcal{A}_{Y_2}\omega Y_1 + C\mathcal{A}_{Y_2}BY_2 + \omega\mathcal{V}\nabla_{Y_2}BY_2\\ & + C\mathcal{H}\nabla_{Y_2}CY_2 + \omega\mathcal{A}_{Y_2}CY_2.\end{aligned}$$

*Proof.* Let $\alpha : I \to N_1$ be a regular curve on $N_1$. Since $Y_1(t)$ and $Y_2(t)$ are the vertical and horizontal parts of the tangent vector field $\dot\alpha(t)$, i.e., $\dot\alpha(t) = Y_1(t) + Y_2(t)$. Using equations (4), (5), (13)-(16), (19), (20) and Lemma 1, we get

$$\begin{aligned}
\nabla_{\dot\alpha}\dot\alpha &= -J(\nabla_{\dot\alpha}J\dot\alpha)\\
&= -J(\nabla_{Y_1}\phi Y_1 + \nabla_{Y_1}\omega Y_1 + \nabla_{Y_2}\phi Y_1 + \nabla_{Y_2}\omega Y_1\\
&\quad + \nabla_{Y_1}BY_2 + \nabla_{Y_1}CY_2 + \nabla_{Y_2}BY_2 + \nabla_{Y_2}CY_2),\\
&= -\nabla_{Y_1}\phi^2 Y_1 - \nabla_{Y_1}\omega\phi Y_1 - \nabla_{Y_2}\phi^2 Y_1 - \nabla_{Y_2}\omega\phi Y_1\\
&\quad - J(\mathcal{T}_{Y_1}\omega Y_1 + \mathcal{H}\nabla_{Y_1}\omega Y_1 + \mathcal{T}_{Y_1}BY_2 + \mathcal{V}\nabla_{Y_2}\omega Y_1 + \mathcal{T}_{Y_1}CY_2\\
&\quad + \mathcal{H}\nabla_{Y_1}CY_2 + \mathcal{H}\nabla_{Y_2}\omega Y_1 + \mathcal{A}_{Y_2}\omega Y_1 + \mathcal{A}_{Y_2}BY_2 + \mathcal{V}\nabla_{Y_2}BY_2\\
&\quad + \mathcal{H}\nabla_{Y_2}CY_2 + \mathcal{A}_{Y_2}CY_2)\\
&= \cos^2\Theta\mathcal{V}\nabla_{\dot\alpha}Y_1 + \cos^2\Theta(\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})Y_1 - \mathcal{T}_{Y_1}\omega\phi Y_1 - \mathcal{H}\nabla_{Y_1}\omega\phi Y_1\\
&\quad - \mathcal{H}\nabla_{Y_2}\omega\phi Y_1 - \mathcal{A}_{Y_2}\omega\phi Y_1 - \phi\mathcal{T}_{Y_1}\omega Y_1 - \omega\mathcal{T}_{Y_1}\omega Y_1 - B\mathcal{H}\nabla_{Y_1}\omega Y_1\\
&\quad - C\mathcal{H}\nabla_{Y_1}\omega Y_1 - B\mathcal{T}_{Y_1}BY_2 - C\mathcal{T}_{Y_1}BY_2 - \phi\mathcal{V}\nabla_{Y_2}\omega Y_1 - \omega\mathcal{V}\nabla_{Y_2}\omega Y_1\\
&\quad - \phi\mathcal{T}_{Y_1}CY_2 - \omega\mathcal{T}_{Y_1}CY_2 - B\mathcal{H}\nabla_{Y_1}CY_2 - C\mathcal{H}\nabla_{Y_1}CY_2 - \mathcal{H}\nabla_{Y_2}\omega Y_1\\
&\quad - C\mathcal{H}\nabla_{Y_2}\omega Y_1 - \phi\mathcal{A}_{Y_2}\omega Y_1 - \omega\mathcal{A}_{Y_2}\omega Y_1 - B\mathcal{A}_{Y_2}BY_2 - C\mathcal{A}_{Y_2}BY_2\\
&\quad - \phi\mathcal{V}\nabla_{Y_2}BY_2 - \omega\mathcal{V}\nabla_{Y_2}BY_2 - B\mathcal{H}\nabla_{Y_2}CY_2 - C\mathcal{H}\nabla_{Y_2}CY_2\\
&\quad - \phi\mathcal{A}_{Y_2}CY_2 - \omega\mathcal{A}_{Y_2}CY_2.
\end{aligned}$$

Taking the vertical and horizontal components in above equation, we get

$$\begin{aligned}
\mathcal{V}\nabla_{\dot\alpha}\dot\alpha &= \cos^2\Theta\mathcal{V}\nabla_{\dot\alpha}Y_1 - \mathcal{T}_{Y_1}\omega\phi Y_1 - \mathcal{A}_{Y_2}\omega\phi Y_1 - \phi\mathcal{T}_{Y_1}\omega Y_1\\
&\quad - B\mathcal{H}\nabla_{Y_1}\omega Y_1 - B\mathcal{T}_{Y_1}BY_2 - \phi\mathcal{V}\nabla_{Y_2}\omega Y_1 - \phi\mathcal{T}_{Y_1}CY_2\\
&\quad - \omega\mathcal{T}_{Y_1}CY_2 - B\mathcal{H}\nabla_{Y_1}CY_2 - \phi\mathcal{A}_{Y_2}\omega Y_1 - B\mathcal{A}_{Y_2}BY_2\\
&\quad - \phi\mathcal{V}\nabla_{Y_2}BY_2 - B\mathcal{H}\nabla_{Y_2}CY_2 - \phi\mathcal{A}_{Y_2}CY_2,
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}\nabla_{\dot\alpha}\dot\alpha &= \cos^2\Theta(\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})Y_1 - \mathcal{H}\nabla_{Y_1}\omega\phi Y_1 - \mathcal{H}\nabla_{Y_2}\omega\phi Y_1 - \omega\mathcal{T}_{Y_1}\omega Y_1\\
&\quad - C\mathcal{H}\nabla_{Y_1}\omega Y_1 - C\mathcal{T}_{Y_1}BY_2 - \omega\mathcal{V}\nabla_{Y_2}\omega Y_1 - \omega\mathcal{T}_{Y_1}CY_2\\
&\quad - C\mathcal{H}\nabla_{Y_1}CY_2 - C\mathcal{H}\nabla_{Y_2}\omega Y_1 - \omega\mathcal{A}_{Y_2}\omega Y_1 - C\mathcal{A}_{Y_2}BY_2\\
&\quad - \omega\mathcal{V}\nabla_{Y_2}BY_2 - C\mathcal{H}\nabla_{Y_2}CY_2 - \omega\mathcal{A}_{Y_2}CY_2.
\end{aligned}$$

Now, $\alpha$ is a geodesic on $N_1$ if and only if $\mathcal{V}\nabla_{\dot\alpha}\dot\alpha = 0$ and $\mathcal{H}\nabla_{\dot\alpha}\dot\alpha = 0$, which is completes proof. □

**Theorem 2.** *Let $F$ be a slant submersion from a Kähler manifold $(N_1, g_1, J)$ onto a Riemannian manifold $(N_2, g_2)$. Then $F$ is a Clairaut slant submersion with $r = e^h$ if and only if*

$$\begin{aligned}
g_1(\mathcal{V}\nabla_{\dot\alpha}\phi Y_1 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})CY_2 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})\omega Y_1, BY_2) + g_1(\mathcal{H}\nabla_{\dot\alpha}\omega Y_1\\
+(\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})BY_2 + (\mathcal{A}_{Y_2} + \mathcal{T}_{Y_1})\phi Y_1, CY_2) + g_1(Y_1, Y_1)\frac{dh}{dt} = 0,
\end{aligned}$$

*where $\alpha : I \to N_1$ is a geodesic on $N_1$ and $Y_1, Y_2$ are vertical and horizontal components of $\dot{\alpha}(t)$.*

*Proof.* Let $\alpha : I \to N_1$ be a geodesic on $N_1$ with $Y_1(t) = \mathcal{V}\dot{\alpha}(t)$ and $Y_2(t) = \mathcal{H}\dot{\alpha}(t)$ denote the angle in $[0, F]$ between $\dot{\alpha}(t)$ and $Y_2(t)$. Assuming $\nu = ||\dot{\alpha}(t)||,^2$ then we get

$$g_1(Y_1(t), Y_1(t)) = \nu \sin^2 \theta(t), \tag{33}$$

$$g_1(Y_2(t), Y_2(t)) = \nu \cos^2 \theta(t). \tag{34}$$

Now, differentiating (33), we get

$$\frac{d}{dt} g_1(Y_1(t), Y_1(t)) = 2\nu \cos\theta(t) \sin\theta(t) \frac{d\theta}{dt},$$

$$\nu \cos\theta(t) \sin\theta(t) \frac{d\theta}{dt} = g_1(\mathcal{V}\nabla_{\dot{\alpha}} Y_1, Y_1),$$

$$\nu \cos^2 \Theta \cos\theta(t) \sin\theta(t) \frac{d\theta}{dt} = g_1(\cos^2 \Theta \mathcal{V}\nabla_{\dot{\alpha}} Y_1, Y_1). \tag{35}$$

On the other hand, using Lemma 4 and equation (35), we get

$$\begin{aligned} &\nu \cos^2 \Theta \cos\theta(t) \sin\theta(t) \frac{d\theta}{dt} \qquad (36) \\ &= g_1(\mathcal{T}_{Y_1}\omega\phi Y_1 + \mathcal{A}_{Y_2}\omega\phi Y_1 + \phi\mathcal{T}_{Y_1}\omega Y_1 + B\mathcal{H}\nabla_{Y_1}\omega Y_1 \\ &\quad + B\mathcal{T}_{Y_1}BY_2 + \phi\mathcal{V}\nabla_{Y_2}\omega Y_1 + \phi\mathcal{T}_{Y_1}CY_2 + \omega\mathcal{T}_{Y_1}CY_2 \\ &\quad + B\mathcal{H}\nabla_{Y_1}CY_2 + \phi\mathcal{A}_{Y_2}\omega Y_1 + B\mathcal{A}_{Y_2}BY_2 + \phi\mathcal{V}\nabla_{Y_2}BY_2 \\ &\quad + B\mathcal{H}\nabla_{Y_2}CY_2 + \phi\mathcal{A}_{Y_2}CY_2, Y_1). \qquad (37) \end{aligned}$$

Moreover, $F$ is a Clairaut slant submersion with $r = e^h$ if and only if $\frac{d}{dt}(e^{h\circ\alpha}\sin\theta) = 0$, i.e., $e^{h\circ\alpha}(\cos\theta\frac{d\theta}{dt} + \sin\theta\frac{dh}{dt}) = 0$. By multiplying this with non-zero factor $\nu\cos^2\Theta\sin\theta$, we have

$$\nu \cos^2 \Theta \cos\theta \sin\theta \frac{d\theta}{dt} = -\nu \cos^2 \Theta \sin^2 \theta \frac{dh}{dt}. \tag{38}$$

Thus, from equations $(17), (36)$ and $(38)$, we have

$$\begin{aligned} &-\cos^2 \Theta ||Y_1||^2 g_1(\nabla h, Y_2) \qquad (39) \\ &= g_1(\mathcal{T}_{Y_1}\omega\phi Y_1 + \mathcal{A}_{Y_2}\omega\phi Y_1 + \phi\mathcal{T}_{Y_1}\omega Y_1 + B\mathcal{H}\nabla_{Y_1}\omega Y_1 \\ &\quad + B\mathcal{T}_{Y_1}BY_2 + \phi\mathcal{V}\nabla_{Y_2}\omega Y_1 + \phi\mathcal{T}_{Y_1}CY_2 + \omega\mathcal{T}_{Y_1}CY_2 \\ &\quad + B\mathcal{H}\nabla_{Y_1}CY_2 + \phi\mathcal{A}_{Y_2}\omega Y_1 + B\mathcal{A}_{Y_2}BY_2 + \phi\mathcal{V}\nabla_{Y_2}BY_2 \\ &\quad + B\mathcal{H}\nabla_{Y_2}CY_2 + \phi\mathcal{A}_{Y_2}CY_2, Y_1), \end{aligned}$$

which is completes the proof. □

## 4. Curvature relations

In this section, we are going to obtain curvature relations of Clairaut slant Riemannian submersions [7].

Let $(N_1, g_1)$ and $(N_2, g_2)$ be two Riemannia manifolds with corresponding curvature relation $R$ and $R^*$, respectively. Let $\pi : (N_1, g_1) \to (N_2, g_2)$ be a Riemannian submersion and $\widehat{R}$ the curvature tensor of fibers of $\digamma$. If $X_1, X_2, X_3, X_4$ are horizontal and $Z_1, Z_2, Z_3, Z_4$ vertical vectors, then

$$R(Z_1, Z_2, Z_3, Z_4) = \widehat{R}(Z_1, Z_2, Z_3, Z_4) - g_1(\mathcal{T}_{Z_1} Z_4, \mathcal{T}_{Z_2} Z_3) + g_1(\mathcal{T}_{Z_2} Z_4, \mathcal{T}_{Z_1} Z_3), \tag{40}$$

$$R(Z_1, Z_2, Z_3, X_1) = g_1((\nabla_{Z_1}\mathcal{T})(Z_2, Z_3), X_1) - g_1((\nabla_{Z_2}\mathcal{T})(Z_1, Z_3), X_1), \tag{41}$$

$$R(X_1, X_2, X_3, Z_1) = -g_1((\nabla_{X_3}\mathcal{A})(X_1, X_2), Z_1) - g_1(\mathcal{A}_{X_1} X_2, \mathcal{T}_{Z_1} X_3) + g_1(\mathcal{A}_{X_2} X_3, \mathcal{T}_{Z_1} X_1) + g_1(\mathcal{A}_{X_3} X_1, \mathcal{T}_{Z_1} X_2), \tag{42}$$

$$R(X_1, X_2, X_3, X_4) = R^*(X_1, X_2, X_3, X_4) + 2g_1(\mathcal{A}_{X_1} X_2, \mathcal{A}_{X_2} X_3) - g_1(\mathcal{A}_{X_2} X_3, \mathcal{A}_{X_1} X_4) + g_1(\mathcal{A}_{X_1} X_3, \mathcal{A}_{X_2} X_4), \tag{43}$$

$$\begin{aligned} R(X_1, X_2, Z_1, Z_2) = &-g_1((\nabla_{Z_1}\mathcal{A})(X_1, X_2), Z_2) \\ &+ g_1((\nabla_{Z_2}\mathcal{A})(X_1, X_2), Z_1) - g_1(\mathcal{A}_{X_1} Z_1, \mathcal{A}_{X_2} Z_2) \\ &+ g_1(\mathcal{A}_{X_1} Z_2, \mathcal{A}_{X_2} Z_1) + g_1(\mathcal{T}_{Z_1} X_1, \mathcal{T}_{Z_2} X_2) \\ &- g_1(\mathcal{T}_{Z_2} X_1, \mathcal{T}_{Z_1} X_2), \end{aligned} \tag{44}$$

$$\begin{aligned} R(X_1, Z_1, X_2, Z_2) = &-g_1((\nabla_{X_1}\mathcal{T})(Z_1, Z_2), X_2) \\ &- g_1((\nabla_{Z_1}\mathcal{T})(X_1, X_2), Z_2) + g_1(\mathcal{T}_{Z_1} X_1, \mathcal{T}_{Z_2} X_2) \\ &- g_1(\mathcal{A}_{X_1} Z_1, \mathcal{A}_{X_2} Z_2). \end{aligned} \tag{45}$$

**Lemma 5.** *Let $\digamma$ be a slant submersion from a Kähler manifold $(N_1, g_1, J)$ onto a Riemannian manifold $(N_2, g_2)$. Then*

$$\begin{aligned} &R(V_1, V_2, V_3, V_4) \\ &= \cos^4\Theta(\widehat{R}(V_1, V_2, V_3, V_4) - g_1(\mathcal{T}_{V_1} V_4, \mathcal{T}_{V_2} V_3) + g_1(\mathcal{T}_{V_2} V_4, \mathcal{T}_{V_1} V_3)) \\ &\quad - g_1((\nabla_{\omega\phi V_2}\mathcal{T})(V_1, V_3), \omega\phi V_4) - g_1((\nabla_{V_1}\mathcal{A})(\omega\phi V_2, \omega\phi V_4), V_3) \\ &\quad + g_1(\mathcal{T}_{V_1}\omega\phi V_2, \mathcal{T}_{V_3}\omega\phi V_4) - g_1(\mathcal{A}_{\omega\phi V_2} V_1, \mathcal{A}_{\omega\phi V_4} V_3) \\ &\quad - \cos^2\Theta g_1((\nabla_{V_1}\mathcal{T})(V_2, V_3), \omega\phi V_4) + \cos^2\Theta g_1((\nabla_{V_2}\mathcal{T})(V_1, V_3), \omega\phi V_4) \\ &\quad + \cos^2\Theta g_1((\nabla_{V_3}\mathcal{T})(V_4, V_1), \omega\phi V_2) - \cos^2\Theta g_1((\nabla_{V_4}\mathcal{T})(V_3, V_1), \omega\phi V_2) \\ &\quad + \cos^2\Theta g_1((\nabla_{V_2}\mathcal{T})(V_1, \phi V_3), \omega V_4) - \cos^2\Theta g_1((\nabla_{V1}\mathcal{T})(V_2, \phi V_3), \omega V_4) \\ &\quad + \cos^2\Theta(-g_1((\nabla_{V_1}\mathcal{A})(\omega V_3, \omega V_4), V_2) \end{aligned}$$

$$
\begin{aligned}
&+ g_1((\nabla_{V_2}\mathcal{A})(\omega V_3, \omega V_4), V_1) - g_1(\mathcal{A}_{\omega V_3} V_1, \mathcal{T}_{V_2}\omega V_4)\\
&+ g_1(\mathcal{A}_{\omega V_3} V_2, \mathcal{T}_{\omega V_4} V_1) + g_1(\mathcal{T}_{V_1}\omega V_3, \mathcal{T}_{V_2}\omega V_4)\\
&- g_1(\mathcal{T}_{V_2}\omega V_3, \mathcal{T}_{V_1}\omega V_4)) + g_1((\nabla_{\omega V_4}\mathcal{T})(\phi V_3, V_1), \omega\phi V_2)\\
&+ g_1((\nabla_{\phi V_3}\mathcal{A})(\omega V_4, \omega\phi V_2), V_1) - g_1(\mathcal{T}_{\phi V_3}\omega V_4, \mathcal{T}_{V_1}\omega\phi V_2)\\
&+ g_1(\mathcal{A}_{\omega V_4}\phi V_3, \mathcal{A}_{\omega\phi V_2} V_1) - g((\nabla_{\omega\phi V_2}\mathcal{A})(\omega V_3, \omega V_4), V_1)\\
&- g_1(A_{\omega V_3}\omega V_4, A_{V_1}\omega\phi V_2) + g_1(\mathcal{A}_{\omega V_4}\omega\phi V_2, \mathcal{A}_{V_1}\omega V_3)\\
&+ g_1(A_{\omega\phi V_2}\omega V_3, T_{V_1}\omega V_4) + \cos^2\Theta(g_1((\nabla_{V_3}\mathcal{T})(V_4, \phi V_1), \omega V_2)\\
&- g_1((\nabla_{V_4}\mathcal{T})(V_3, \phi V_1), \omega V_2)) + \cos^2\Theta(-g((\nabla_{V_3}\mathcal{A})(\omega V_1, \omega V_2), V_4)\\
&+ g_1((\nabla_{V_4} A)(\omega V_1, \omega V_2), V_3) - g_1(\mathcal{A}_{\omega V_1} V_3, \mathcal{A}_{\omega V_2} V_4)\\
&+ g_1(\mathcal{A}_{\omega V_1} V_4, \mathcal{A}_{\omega V_2} V_3) + g_1(\mathcal{T}_{V_3}\omega V_1, \mathcal{T}_{V_4}\omega V_2)\\
&- g_1(\mathcal{T}_{V_4}\omega V_1, \mathcal{T}_{V_3}\omega V_2)) + g_1((\nabla_{\omega V_2}\mathcal{T})(\phi V_1, V_3), \omega\phi V_4)\\
&+ g_1((\nabla_{\phi V_1}\mathcal{A})(\omega V_2, \omega\phi V_2), V_3) - g_1(\mathcal{T}_{\phi V_1}\omega V_2, \mathcal{T}_{V_3}\omega\phi V_2)\\
&+ g_1(\mathcal{A}_{\omega V_2}\phi V_1, \mathcal{A}_{\omega\phi V_4} Z) - g_1((\nabla_{\omega\phi V_4}\mathcal{A})(\omega V_1, \omega V_2), V_3)\\
&- g_1(\mathcal{A}_{\omega V_1}\omega V_2, \mathcal{T}_{V_3}\omega\phi V_4) + g_1(\mathcal{A}_{\omega V_2}\omega\phi V_4, \mathcal{T}_{V_3}\omega V_1)\\
&+ g_1(\mathcal{A}_{\omega\phi V_4}\omega V_1, \mathcal{T}_{V_3}\omega V_2) - g_1((\nabla_{\omega V_2}\mathcal{T})(\phi V_1, \phi V_3), \omega V_4)\\
&- g_1((\nabla_{\phi V_1}\mathcal{A})(\omega V_2, \omega V_4), \phi V_3) + g_1(\mathcal{T}_{\phi V_1}\omega V_2, \mathcal{T}_{\phi V_3}\omega V_4)\\
&- g_1(\mathcal{A}_{\omega V_2}\phi V_1, \mathcal{A}_{\omega V_4}\phi V_3) + g((\nabla_{\omega V_2}\mathcal{A})(\omega V_3, \omega V_4), \phi V_1)\\
&+ g_1(\mathcal{A}_{\omega V_3}\omega V_4, \mathcal{T}_{\phi V_3}\omega V_2) - g_1(\mathcal{A}_{\omega V_4}\omega V_2, \mathcal{T}_{\phi V_1}\omega V_3)\\
&- g_1(\mathcal{A}_{\omega V_2}\omega V_3, \mathcal{T}_{\phi V_1}\omega V_4) + g((\nabla_{\omega V_4}\mathcal{A})(\omega V_1, \omega V_2), \phi V_3)\\
&+ g_1(\mathcal{A}_{\omega V_1}\omega V_2, \mathcal{T}_{\phi V_3}\omega V_4) - g_1(\mathcal{A}_{\omega V_2}\omega V_4, \mathcal{T}_{\phi V_3}\omega V_1)\\
&- g_1(\mathcal{A}_{\omega V_2}\omega V_2, \mathcal{T}_{\phi V_3}\omega V_2) + R^*(\omega V_1, \omega V_2, \omega V_3, \omega V_4)\\
&+ 2g_1(\mathcal{A}_{\omega V_1}\omega V_2, \mathcal{A}_{\omega V_3}\omega V_4) - g(\mathcal{A}_{\omega V_2}\omega V_3, \mathcal{A}_{\omega V_1}\omega V_4)\\
&+ g(A_{\omega V_1}\omega V_3, A_{\omega V_2}\omega V_4),
\end{aligned}
$$

$$
\begin{aligned}
&R(Z_1, Z_2, Z_3, Z_4)\\
&= \widehat{R}(BZ_1, BZ_2, BZ_3, BZ_4) - g_1(\mathcal{T}_{BZ_1} BZ_4, \mathcal{T}_{BZ_2} BZ_3)\\
&\quad + g_1(\mathcal{T}_{BZ_2} BZ_4, \mathcal{T}_{BZ_1} BZ_3) + g_1((\nabla_{BZ_1}\mathcal{T})(BZ_2, BZ_3), CZ_4)\\
&\quad - g_1((\nabla_{BZ_2}\mathcal{T})(BZ_1, BZ_3), CZ_4) - g_1((\nabla_{BZ_1}\mathcal{T})(BZ_2, BZ_4), CZ_3)\\
&\quad + g_1((\nabla_{BZ_2}\mathcal{T})(BZ_1, BZ_4), CZ_3) - g_1((\nabla_{BZ_1}\mathcal{A})(CZ_3, CZ_4), BZ_2)\\
&\quad + g_1((\nabla_{BZ_2}\mathcal{A})(CZ_3, CZ_4), BZ_1) + g_1(\mathcal{A}_{CZ_3} BZ_1, \mathcal{A}_{CZ_4} BZ_2)\\
&\quad - g_1(\mathcal{A}_{CZ_3} BZ_2, \mathcal{A}_{CZ_4} BZ_1) + g_1(\mathcal{T}_{BZ_1} CZ_3, \mathcal{T}_{BZ_2} CZ_4)\\
&\quad - g_1(\mathcal{T}_{BZ_2} CZ_3, \mathcal{T}_{BZ_1} CZ_4) + g_1((\nabla_{BZ_3}\mathcal{T})(BZ_4, BZ_1), CZ_2)\\
&\quad - g_1((\nabla_{BZ_4}\mathcal{T})(BZ_3, BZ_1), CZ_2) - g_1((\nabla_{CZ_2}\mathcal{T})(BZ_1, BZ_3), CZ_4)\\
&\quad - g_1((\nabla_{BZ_1}\mathcal{A})(CZ_2, CZ_4), BZ_3) + g_1(\mathcal{T}_{BZ_1} CZ_2, \mathcal{T}_{BZ_3} CZ_4)
\end{aligned}
$$

$$
\begin{aligned}
&- g_1(\mathcal{A}_{CZ_2}BZ_1, \mathcal{A}_{CZ_4}BZ_3) + g_1((\nabla_{CZ_2}\mathcal{T})(BZ_1, BZ_4), CZ_3)\\
&+ g_1((\nabla_{BZ_1}\mathcal{A})(CZ_2, CZ_3), BZ_4) - g_1(\mathcal{T}_{BZ_1}CZ_2, \mathcal{T}_{BZ_4}CZ_3)\\
&- g_1(\mathcal{A}_{CZ_2}BZ_1, \mathcal{A}_{CZ_3}BZ_4) + g_1((\nabla_{CZ_2}\mathcal{A})(CZ_3, CZ_4), BZ_1)\\
&+ g_1(\mathcal{A}_{CZ_1}CZ_4, \mathcal{T}_{BZ_1}CZ_2) - g_1(\mathcal{A}_{CZ_4}CZ_2, \mathcal{T}_{BZ_1}CZ_3)\\
&- g_1(\mathcal{A}_{CZ_2}CZ_3, \mathcal{T}_{BZ_1}CZ_4) - g_1((\nabla_{BZ_3}\mathcal{T})(BZ_4, BZ_2), CZ_1)\\
&+ g_1((\nabla_{BZ_4}\mathcal{T})(BZ_3, BZ_2), CZ_1) + g_1((\nabla_{CZ_1}\mathcal{T})(BZ_2, BZ_3), CZ_4)\\
&+ g_1((\nabla_{BZ_2}\mathcal{A})(CZ_1, CZ_4), BZ_3) - g_1(\mathcal{T}_{BZ_2}CZ_1, \mathcal{T}_{BZ_3}CZ_4)\\
&+ g_1(\mathcal{A}_{CZ_1}BZ_2, \mathcal{A}_{CZ_4}BZ_3) + g_1((\nabla_{CZ_1}\mathcal{T})(BZ_2, BZ_3), CZ_4)\\
&+ g_1((\nabla_{BZ_2}\mathcal{A})(CZ_1, CZ_4), BZ_3) - g_1(\mathcal{T}_{BZ_2}CZ_1, \mathcal{T}_{BZ_3}CZ_4)\\
&+ g_1(\mathcal{A}_{CZ_1}BZ_2, \mathcal{A}_{CZ_4}BZ_3) - g_1((\nabla_{CZ_1}\mathcal{A})(CZ_3, CZ_4), BZ_2)\\
&- g_1(\mathcal{A}_{CZ_3}CZ_4, \mathcal{T}_{BZ_2}CZ_1) + g_1(\mathcal{A}_{CZ_4}CZ_1, \mathcal{T}_{BZ_2}CZ_3)\\
&+ g_1(\mathcal{A}_{CZ_1}CZ_3, \mathcal{T}_{BZ_2}CZ_4) - g_1((\nabla_{BZ_3}\mathcal{A})(CZ_1, CZ_2), BZ_4)\\
&+ g_1((\nabla_{BZ_4}\mathcal{A})(CZ_1, CZ_2), BZ_3) - g_1(\mathcal{A}_{CZ_1}BZ_3, \mathcal{A}_{CZ_2}BZ_4)\\
&+ g_1(\mathcal{A}_{CZ_1}BZ_4, \mathcal{A}_{CZ_2}BZ_3) + g_1(\mathcal{T}_{BZ_3}CZ_1, \mathcal{T}_{BZ_4}CZ_2)\\
&- g_1(\mathcal{T}_{BZ_4}CZ_1, \mathcal{T}_{BZ3}CZ_2) + g_1((\nabla_{CZ_4}\mathcal{A})(CZ_1, CZ_2), BZ_3)\\
&+ g_1(\mathcal{A}_{CZ_1}CZ_2, \mathcal{T}_{BZ_3}CZ_4) - g_1(\mathcal{A}_{CZ_2}CZ_4, \mathcal{T}_{BZ_3}CZ_1)\\
&+ g_1(\mathcal{A}_{CZ_4}CZ_1, \mathcal{T}_{BZ_3}CZ_2) - g_1((\nabla_{CZ_3}\mathcal{A})(CZ_1, CZ_2), BZ_4)\\
&- g_1(\mathcal{A}_{CZ_1}CZ_2, \mathcal{T}_{BZ_4}CZ_3) + g_1(\mathcal{A}_{CZ_2}CZ_3, \mathcal{T}_{BZ_4}CZ_1)\\
&+ g_1(\mathcal{A}_{CZ_3}CZ_2, \mathcal{T}_{BZ_4}CZ_2) + R^*(CZ_1, CZ_2, CZ_3, CZ_4)\\
&+ 2g_1(\mathcal{A}_{CZ_1}CZ_2, \mathcal{A}_{CZ_3}CZ_4) - g_1(\mathcal{A}_{CZ_2}CZ_3, \mathcal{A}_{CZ_1}CZ_4)\\
&+ g_1(\mathcal{A}_{CZ_1}CZ_3, \mathcal{A}_{CZ_2}CZ_4),
\end{aligned}
$$

$$
\begin{aligned}
&R(Z_1, V_1, Z_2, V_2)\\
&= \cos^4\Theta(-g_1((\nabla_{Z_1}\mathcal{T})(V_1, V_2), Z_2) - g_1((\nabla_{V_1}\mathcal{A})(Z_1, Z_2), V_2)\\
&+ g_1(\mathcal{T}_{V_1}Z_1, \mathcal{T}_{V_2}Z_2) - g_1(\mathcal{A}_{Z_1}V_1, \mathcal{A}_{Z_2}V_2))\\
&- \cos^2\Theta(-g_1((\nabla_{Z_1}\mathcal{A})(Z_2, \omega\phi V_2), V_1) - g_1(\mathcal{A}_{Z_2}\omega\phi V_2, \mathcal{T}_{V_1}Z_1)\\
&+ g_1(\mathcal{A}_{\omega\phi V_2}Z_1, \mathcal{T}_{V_1}Z_2) + g_1(\mathcal{A}_{Z_1}Z_2, \mathcal{T}_{V_1}\omega\phi V_2))\\
&+ R^*(Z_1, \omega\phi V_1, Z_2, \omega\phi V_2) + 2g_1(\mathcal{A}_{Z_1}\omega\phi V_1, \mathcal{A}_2\omega\phi V_2)\\
&- g_1(\mathcal{A}_{\omega\phi V_1}Z_2, \mathcal{A}_{Z_1}\omega\phi V_2) + g(\mathcal{A}_{Z_1}Z_2, \mathcal{A}_{\omega\phi V_1}\omega\phi V_2)\\
&- \cos^2(-g_1(\nabla_{Z_2}\mathcal{A})(Z_1, \omega\phi V_1), V_2) - g_1(\mathcal{A}_{Z_1}\omega\phi V_1, \mathcal{T}_{V_2}Z_2)\\
&+ g_1(\mathcal{A}_{\omega\phi V_1}Z_2, \mathcal{T}_{V_2}Z_1) + g_1(\mathcal{A}_{Z_2}Z_1, \mathcal{T}_{V_2}\omega\phi V_1))\\
&- \cos^2\Theta(-g_1((\nabla_{\omega V_2}\mathcal{T})(BZ_2, V_1), Z_1) - g_1((\nabla_{BZ_2}\mathcal{A})(\omega V_2, Z_1), V_1)\\
&+ g_1(\mathcal{T}_{BZ_2}\omega V_2, \mathcal{T}_{V_1}Z_1) - g_1(\mathcal{A}_{\omega V_2}BZ_2, \mathcal{A}_{Z_1}V_1))\\
&+ \cos^2\Theta(-g_1((\nabla_{Z_1}\mathcal{A})(CZ_2, \omega V_2), V_1) - g_1(\mathcal{T}_{CZ_2}\omega V_2, \mathcal{T}_{V_1}Z_1)
\end{aligned}
$$

$$
\begin{aligned}
&+ g_1(\mathcal{A}_{\omega V_2}Z_1, \mathcal{T}_{V_1}CZ_2) + g_1(\mathcal{A}_{Z_1}CZ_2, \mathcal{T}_{V_1}\omega V_2)\\
&- g_1((\nabla_{\omega V_2}\mathcal{A})(Z_1, \omega\phi V_1), BZ_2) - g_1(\mathcal{A}_{Z_1}\omega\phi V_1, \mathcal{T}_{BZ_2}\omega V_2)\\
&+ g_1(\mathcal{A}_{\omega\phi V_1}\omega V_2, \mathcal{T}_{BZ_2}Z_1) + g_1(\mathcal{A}_{\omega V_2}Z_1, \mathcal{T}_{BZ_2}\omega\phi V_1)\\
&- R^*(CZ_2, \omega V_2, Z_1, \omega\phi V_1) - 2g_1(\mathcal{A}_{CZ_2}\omega V_2, \mathcal{A}_{Z_1}\omega\phi V_1)\\
&- g_1(\mathcal{A}_{\omega V_2}Z_1, \mathcal{A}_{CZ_2}\omega\phi V_1) + g_1(\mathcal{A}_{CZ_2}Z_1, \mathcal{A}_{\omega V_2}\omega\phi V_1)\\
&- \cos^2\Theta(-g_1((\nabla_{\omega V_1}\mathcal{T})(BZ_1, V_2), Z_2) - g_1((\nabla_{V_2}\mathcal{A})(\omega V_1, Z_2), BZ_1)\\
&+ g_1(\mathcal{T}_{V_2}Z_2, \mathcal{T}_{BZ_1}\omega V_1) - g_1(\mathcal{A}_{\omega V_1}Z_2, \mathcal{A}_{BZ_1}V_2))\\
&+ \cos^2\Theta(-g_1((\nabla_{Z_2}\mathcal{A})(CZ_1, \omega V_1), V_2) - g_1(\mathcal{A}_{CZ_1}\omega V_1, \mathcal{T}_{V_2}Z_2)\\
&+ g_1(\mathcal{A}_{\omega V_1}Z_2, \mathcal{T}_{V_2}CZ_1) + g_1(\mathcal{A}_{Z_2}CZ_1, \mathcal{T}_{V_2}\omega V_1))\\
&- g_1((\nabla_{\omega V_1}\mathcal{A})(Z_2, \omega\phi V_2), BZ_1) - g_1(\mathcal{A}_{Z_2}\omega\phi V_2, \mathcal{T}_{BZ_1}\omega V_1)\\
&+ g_1(\mathcal{A}_{\omega\phi V_2}\omega V_1, \mathcal{T}_{BZ_1}Z_2) + g_1(\mathcal{A}_{\omega V_1}Z_2, \mathcal{T}_{BZ_1}\omega\phi V_2)\\
&- R^*(CZ_1, \omega V_1, Z_2, \omega\phi V_2) + 2g_1(\mathcal{A}_{CZ_1}\omega V_1, \mathcal{A}_{Z_2}\omega\phi V_2)\\
&- g_1(\mathcal{A}_{\omega V_1}Z_2, \mathcal{A}_{CZ_2}\omega\phi V_2) + g_1(\mathcal{A}_{CZ_1}Z_2, \mathcal{A}_{\omega V_1}\omega\phi V_2)\\
&- g_1((\nabla_{\omega V_1}\mathcal{T})(BZ_1, BZ_2), \omega V_1) - g_1((\nabla_{BZ_1}\mathcal{A})(\omega V_1, \omega V_2), BZ_2)\\
&+ g_1(\mathcal{T}_{BZ_1}\omega V_1, \mathcal{T}_{BZ_2}\omega V_2) - g_1(\mathcal{A}_{\omega V_1}BZ_1, \mathcal{A}_{\omega V_2}BZ_2)\\
&+ g_1((\nabla_{\omega V_1}\mathcal{T})(CZ_1, \omega V_2), BZ_1) + g_1(\mathcal{A}_{CZ_2}\omega V_2, \mathcal{T}_{BZ_1}\omega V_1)\\
&- g_1(\mathcal{A}_{\omega V_2}\omega V_1, \mathcal{T}_{BZ_1}CZ_2) - g_1(\mathcal{A}_{\omega V_1}CZ_2, \mathcal{T}_{BZ_1}\omega V_2)\\
&+ g_1((\nabla_{\omega V_2}\mathcal{T})(CZ_2, \omega V_1), BZ_2) + g_1(\mathcal{A}_{CZ_1}\omega V_1, \mathcal{T}_{BZ_2}\omega V_2)\\
&- g_1(\mathcal{A}_{\omega V_1}\omega V_2, \mathcal{T}_{BZ_2}CZ_1) - g_1(\mathcal{A}_{\omega V_2}CZ_1, \mathcal{T}_{BZ_2}\omega V_1)\\
&+ R^*(CZ_1, \omega V_1, CZ_2, \omega V_1) + 2g_1(\mathcal{A}_{CZ_1}\omega V_1, \mathcal{T}_{CZ_2}\omega V_2)\\
&- g_1(\mathcal{A}_{\omega V_1}CZ_2, \mathcal{A}_{CZ_1}\omega V_2) + g_1(\mathcal{A}_{CZ_1}CZ_2, \mathcal{A}_{\omega V_1}\omega V_2).
\end{aligned}
$$

*for* $Z_1, Z_2 \in \Gamma(\ker F_*)^\perp$ *and* $V_1, V_2 \in \Gamma(\ker F_*)$.

*Proof.* Using equations (6)-(10), (40)-(45) and Lemma 1, we can easily get Lemma 5. □

## 5. Example

**Example 1.** Let $N_1$ be an Euclidean space given by

$$N_1 = \{(x_1, x_2, x_3, x_4) \in R^4 : (x_1, x_2, x_3, x_4) \neq (0,0,0,0)\}.$$

We define the Riemannian metric $g_1$ on $N_1$ given by

$$g_1 = e^{2x_4}dx_1^2 + e^{2x_4}dx_2^2 + e^{2x_4}dx_3^2 + dx_4^2$$

and the complex structure on $J$ and $N_1$ defined as

$$J(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3).$$

Let $N_2 = \{(v_1, v_2, v_3) \in R^3\}$ be a Riemannian manifold with Riemannian metric $g_2$ on $N_2$ given by $g_2 = e^{2x_4}dv_1^2 + dv_2^2$.

Define a map $F : R^4 \to R^2$ by $F(x_1, x_2, x_3, x_4) = (\frac{x_1 - x_3}{\sqrt{2}}, x_4)$. Then we have

$$(\ker F_*) = \langle X_1 = (e_1 + e_3), X_2 = e_2 \rangle,$$

and

$$(\ker F_*)^\perp = \langle V_1 = (e_1 - e_3), V_2 = e_4 \rangle,$$

where

$$\Big\{ e_1 = e^{-x_4} \frac{\partial}{\partial x_1}, \quad e_2 = e^{-x_4} \frac{\partial}{\partial x_2}, \quad e_3 = e^{-x_4} \frac{\partial}{\partial x_3}, \quad e_4 = \frac{\partial}{\partial x_4} \Big\},$$

$$\Big\{ e_1^* = \frac{\partial}{\partial v_1}, \quad e_2^* = \frac{\partial}{\partial v_2} \Big\}$$

are bases on $T_q N_1$ and $T_{F(q)} N_2$ respectively, for all $p \in N_1$. By direct computations, we can see that

$$F_*(V_1) = \sqrt{2} e^{-x_4} e_1^*, F_*(V_2) = e_2^*$$

and

$$g_1(V_i, V_j) = g_2(F_* V_i, F_* V_j),$$

for all $V_i, V_j \in \Gamma(\ker F_*)^\perp$, $i = 1, 2$. Therefore $F$ is a slant submersioon with slant angle $\Theta = \frac{\pi}{4}$.

Now, we will find smooth function $h$ on $N_1$ satisfying $\mathcal{T}_X X = g_1(X, X) \nabla f$, for all $X \in \Gamma(\ker F_*)$. Since covariant derivative for vector fields $E = E_i \frac{\partial}{\partial x_i}$, $F = F_j \frac{\partial}{\partial x_j}$ on $N_1$ is defined as

$$\nabla_E F = E_i F_j \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + E_i \frac{\partial F_j}{\partial x_i} \frac{\partial}{\partial x_j}, \tag{46}$$

where the covariant derivative of basis vector fields $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial x_i}$ is defined by

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \Gamma_{ij}^k \frac{\partial}{\partial x_k}, \tag{47}$$

and Christoffel symbols are defined by

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left( \frac{\partial g_{1jl}}{\partial x_i} + \frac{\partial g_{1il}}{\partial x_j} - \frac{\partial g_{1ij}}{\partial x_k} \right). \tag{48}$$

Now, we get

$$g_{1ij} = \begin{bmatrix} e^{2x_4} & 0 & 0 & 0 \\ 0 & e^{2x_4} & 0 & 0 \\ 0 & 0 & e^{2x_4} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad g_1^{ij} = \begin{bmatrix} e^{-2x_4} & 0 & 0 & 0 \\ 0 & e^{-2x_4} & 0 & 0 \\ 0 & 0 & e^{-2x_4} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{49}$$

By using (48) and (49), we get

$$\begin{aligned} &\Gamma_{11}^1 = 0, \quad \Gamma_{11}^2 = 0, \quad \Gamma_{11}^3 = 0, \quad \Gamma_{11}^4 = -e^{-2x_4}, \\ &\Gamma_{22}^1 = 0, \quad \Gamma_{22}^2 = 0, \quad \Gamma_{22}^3 = 0, \quad \Gamma_{22}^4 = -e^{-2x_4}, \end{aligned} \tag{50}$$

$$\begin{aligned}
&\Gamma^1_{33} = 0, \quad \Gamma^2_{33} = 0, \quad \Gamma^3_{33} = 0, \quad \Gamma^4_{33} = -e^{-2x_4},\\
&\Gamma^1_{12} = \Gamma^2_{12} = \Gamma^3_{12} = \Gamma^4_{12} = 0,\\
&\Gamma^1_{21} = \Gamma^2_{21} = \Gamma^3_{21} = \Gamma^4_{21} = 0,\\
&\Gamma^1_{13} = \Gamma^2_{13} = \Gamma^3_{13} = \Gamma^4_{13} = 0,\\
&\Gamma^1_{31} = \Gamma^2_{31} = \Gamma^3_{31} = \Gamma^4_{31} = 0,\\
&\Gamma^1_{23} = \Gamma^2_{23} = \Gamma^3_{23} = \Gamma^4_{23} = 0,\\
&\Gamma^1_{32} = \Gamma^2_{32} = \Gamma^3_{32} = \Gamma^4_{32} = 0.
\end{aligned}$$

Using equations (47) and (50), we obtain

$$\begin{aligned}
&\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = -\frac{\partial}{\partial x_4},\\
&\nabla_{e_1} e_2 = \nabla_{e_1} e_3 = \nabla_{e_2} e_1 = \nabla_{e_2} e_3 = 0,\\
&\nabla_{e_3} e_1 = \nabla_{e_3} e_2 = 0,
\end{aligned} \tag{51}$$

Therefore

$$\nabla_{X_1} X_1 = \nabla_{e_1+e_3} e_1 + e_3 = -2\frac{\partial}{\partial x_4}, \nabla_{X_2} X_2 = \nabla_{e_2} e_2 = -\frac{\partial}{\partial x_4}. \tag{52}$$

Now, we have

$$T_X X = T_{\lambda_1 X_1 + \lambda_2 X_2} \lambda_1 X_1 + \lambda_2 X_2, \lambda_1, \lambda_2 \in R,$$

$$T_X X = \lambda_1^2 T_{X_1} X_1 + \lambda_2^2 T_{X_2} X_2 + 2\lambda_1 \lambda_2 T_{X_1} X_2. \tag{53}$$

Using (52), we obtain

$$T_{X_1} X_1 = -2\frac{\partial}{\partial x_4}, T_{X_2} X_2 = -\frac{\partial}{\partial x_4}, T_{X_1} X_2 = 0. \tag{54}$$

Next, using (53) and (54), we get

$$\mathcal{T}_X X = -(2\lambda_1^2 + \lambda_2^2)\frac{\partial}{\partial x_4}. \tag{55}$$

Since $X = \lambda_1 X_1 + \lambda_2 X_2$, so

$$g_1(\lambda_1 X_1 + \lambda_2 X_2, \lambda_1 X_1 + \lambda_2 X_2) = 2\lambda_1^2 + \lambda_2^2.$$

For any smooth function $h$ on $R^4$, the gradient of $h$ with respect to the metric $g_1$ is given by

$$\nabla h = \sum_{i,j=1}^{4} g_1^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}.$$

Hence

$$\nabla h = e^{-2x_4} \frac{\partial h}{\partial x_1} \frac{\partial}{\partial x_1} + e^{-2x_4} \frac{\partial h}{\partial x_2} \frac{\partial}{\partial x_2} + e^{-2x_4} \frac{\partial h}{\partial x_3} \frac{\partial}{\partial x_3} + \frac{\partial h}{\partial x_4} \frac{\partial}{\partial x_4}.$$

Hence $\nabla h = \frac{\partial}{\partial x_4}$ for the function $h = x_4$. Then it is easy to see that $T_X X = -g_1(X,X)\nabla h$, thus by **Theorem 1**, is a Clairaut slant Riemannian submersion.

**Example 2.** Let $N_1$ be an Euclidean space given by

$$N_1 = \{(x_1, x_2, x_{3,} x_4) \in R^4 : (x_1, x_2, x_{3,} x_4) \neq (0,0,0,0)\}.$$

We define the Riemannian metric $g_1$ on $N_1$ given by

$$g_1 = e^{2x_4}dx_1^2 + e^{2x_4}dx_2^2 + e^{2x_4}dx_3^2 + dx_4^2$$

and the complex structure on $J$ and $N_1$ defined as

$$J(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3).$$

Let $N_2 = \{(v_1, v_2, v_3) \in R^3\}$ be a Riemannian manifold with Riemannian metric $g_2$ on $N_2$ given by $g_2 = e^{2x_4}du_1^2 + du_2^2$.

Define a map $\digamma : R^4 \to R^2$ by $\digamma(x_1, x_2, x_3, x_4) = (\frac{x_2 - \sqrt{3}x_3}{2}, x_4)$. Then we have

$$(\ker \digamma_*) = \langle V_1 = e_1, V_2 = \sqrt{3}e_2 + e_3 \rangle,$$

and

$$(\ker \digamma_*)^\perp = \langle H_1 = e_2 - \sqrt{3}e_3, H_2 = e_4 \rangle,$$

where

$$\left\{ e_1 = e^{-x_4}\frac{\partial}{\partial x_1}, \quad e_2 = e^{-x_4}\frac{\partial}{\partial x_2}, \quad e_3 = e^{-x_4}\frac{\partial}{\partial x_3}, \quad e_4 = \frac{\partial}{\partial x_4} \right\},$$

$$\left\{ e_1^* = \frac{\partial}{\partial u_1}, \quad e_2^* = \frac{\partial}{\partial u_2} \right\}$$

are bases on $T_q N_1$ and $T_{\digamma(q)} N_2$ respectively, for all $p \in N_1$. By direct computations, we can see that

$$\digamma_*(H_1) = 2e^{-x_4}\frac{\partial}{\partial u_1}, \digamma_*(H_2) = \frac{\partial}{\partial u_1}$$

and

$$g_1(H_i, H_j) = g_2(\digamma_* H_i, \digamma_* H_j),$$

for all $H_i, H_j \in \Gamma(\ker \digamma_*)^\perp$, $i = 1, 2$. Therefore $\digamma$ is a slant submersion with slant angle $\Theta = \frac{\pi}{6}$.

Now, we will find smooth function $R^4$ on satisfying $\mathcal{T}_V V = g_1(V,V)\nabla h$ for all $V \in \Gamma(\ker \digamma_*)$.

Using the given complex structure, we find

$$\begin{aligned} &[e_1, e_1] = [e_2, e_2] = [e_3, e_3] = [e_4, e_4] = 0, \\ &[e_1, e_2] = 0, [e_1, e_3] = 0, [e_1, e_4] = e_1, \\ &[e_2, e_3] = 0, [e_2, e_4] = e_2, [e_3, e_4] = e_3. \end{aligned} \tag{56}$$

The Levi-Civita connection $\nabla$ of the metric $g_1$ is given by the Koszul's formula, which is

$$(57)\quad \begin{aligned} 2g_1(\nabla_X Z, V) &= Xg_1(Z,V) + Zg_1(V,X) - Vg_1(X,Z) \\ &\quad - g_1([X,Z],V) - g_1([Z,V],X) + g_1([V,X],Z). \end{aligned}$$

Using (56) and (57), we have

$$(58)\quad \begin{aligned} &\nabla_{e_1}e_1 = -e_4, \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_3 = 0, \quad \nabla_{e_2}e_1 = 0, \quad \nabla_{e_2}e_2 = -e_4, \\ &\nabla_{e_2}e_3 = 0, \quad \nabla_{e_3}e_1 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_3 = -e_4. \end{aligned}$$

Therefore

$$(59)\quad \begin{aligned} &\nabla_{V_1}V_1 = \nabla_{e_1}e_1 = -e_4, \quad \nabla_{V_1}V_2 = \nabla_{e_1}\sqrt{3}e_2 + e_3 = 0 \\ &\nabla_{V_2}V_1 = \nabla_{\sqrt{3}e_2+e_3}e_1 = 0, \quad \nabla_{\sqrt{3}e_2+e_3}\sqrt{3}e_2 + e_3 = -4e_4. \end{aligned}$$

Now, we have

$$\mathcal{T}_V V = \nabla_{\lambda_1 V_1 + \lambda_2 V_2}\lambda_1 V_1 + \lambda_2 V_2, \lambda_1\lambda_2 \in R.$$

$$(60)\quad \mathcal{T}_V V = \lambda_1^2 \mathcal{T}_{V_1}V_1 + \lambda_2^2 \mathcal{T}_{V_2}V_2 + \lambda_1\lambda_2\mathcal{T}_{V_1}V_2 + \lambda_1\lambda_2\mathcal{T}_{V_2}V_1.$$

Using (59), we obtain

$$(61)\quad \mathcal{T}_{V_1}V_1 = -e_4, \mathcal{T}_{V_1}V_2 = 0, \mathcal{T}_{V_2}V_1 = 0, \quad \mathcal{T}_{V_2}V_2 = -4e_4.$$

Using (60) and (61), we get

$$(62)\quad \mathcal{T}_V V = -(\lambda_1^2 + 4\lambda_2^2)\frac{\partial}{\partial x_4}.$$

Since $V = \lambda_1 V_1 + \lambda_2 V_2$, so

$$g_1(\lambda_1 V_1 + \lambda_2 V_2, \lambda_1 V_1 + \lambda_2 V_2) = \lambda_1^2 + 4\lambda_2^2.$$

For any smooth function $h$ on $R^4$ the gradient of $h$ with respect to the metric $g_1$ is given by

$$\nabla h = e^{-2x_4}\frac{\partial h}{\partial x_1}\frac{\partial}{\partial x_1} + e^{-2x_4}\frac{\partial h}{\partial x_2}\frac{\partial}{\partial x_2} + e^{-2x_4}\frac{\partial h}{\partial x_3}\frac{\partial}{\partial x_3} + \frac{\partial h}{\partial x_4}\frac{\partial}{\partial x_4}.$$

Hence $\nabla h = \frac{\partial}{\partial x_4}$ for the function $h = x_4$. Then it is easy to see that $\mathcal{T}_V V = g_1(V,V)\nabla h$, thus by **Theorem 1**, it is a Clairaut slant Riemannian submersion.

## 6. CONCLUSION

We introduce Clairaut slant submersions from Almost Hermitian manifolds onto Riemannian manifolds in the present paper. We discuss the geomtrical properties of Clairaut slant submersions from Kähler manifolds onto Riemannian manifolds. With the help of Theorem 1, we prove that $F$ is a Clairaut slant Riemannian submersion in Euclidean space with almost complex structure. Finally, the submersion with almost contact structure in Euclidean space is investigated.

## References

[1] D. Allison, *Lorentzian Clairaut submersions*, Geometriae Dedicata, 63 (3) (1996), 309-319.

[2] C. Altafini, *Redundand robotic chains on Riemannian submersions*, IEEE Transaction on Robotics and Automation, 20 (2) (2004), 335-340.

[3] K. Aso, S. Yorozu, *A generalization of Clairaut's theorem and umbilic foliations*, Nihonkai Mathematical Journal, 2 (2) (1991), 139-153.

[4] RL. Bishop, *Clairaut submersions. Differential Geometry*, (In Honor of Kentaro Yano) Tokyo, Kinokuniya, (1991), 21-31.

[5] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, 203 (2002), Boston, Basel, Berlin.

[6] J. P. Bourguignon, H. B. Lawson, *A mathematician's visit to Kaluza-Klein theory*, Rendiconti del Seminario Matematico Università e Politecnico di Torino, Special Issue (1989), 143-163.

[7] M. Falcitelli, S. Ianus, A. M. Pastore, *Riemannian Submersions and Related Topics*, World Scientific, (2004).

[8] A. Gray, *Pseudo-Riemannian almost product manifolds and submersions*, Journal of Mathematics and Mechanics, 16 (1967), 715-737.

[9] S. Ianus, M. Visinescu, *Space-time compactication and Riemannian submersions*, In: The Mathematical Heritage of C. F. Gauss, ed. G. R assias (World Scientific, River Edge) (1991), 358-371.

[10] S. Kumar, A. T. Vanli, S. Kumar, R. Prasad, *Conformal quasi bi-slant submersions*, Annals of the Alexandru Ioan Cuza University Mathematics, 68 (2) (2022), 167-184.

[11] S. Kumar, R. Prasad, *Conformal anti-invariant submersions from Kenmotsu manifolds onto Riemannian manifolds*, Italian Journal of Pure and Applied Mathematics, (40) (2018), 474-500.

[12] S. Kumar, R. Prasad, S. Kumar, *Clairaut semi-invariant Riemannian maps from almost Hermitian manifolds*, Turkish Journal of Mathematics, 46 (4) (2022), 1193-1209.

[13] S. Kumar, A. K. Rai, R. Prasad, *Pointwise slant submersions from Kenmotsu manifolds into Riemannian manifolds*, Italian Journal of Pure and Applied Mathematics, 38 (2017), 561-572.

[14] J. Lee, J. H. Park, B. Şahin, D. Y. Song, *Einstein conditions for the base of anti-invariant Riemannian submersions and Clairaut submersions*, Taiwanese Journal of Mathematics, 19 (4) (2015), 1145-1160.

[15] B. O'Neill, *The fundamental equations of a submersion*, Michigan Mathematical Journal, 13 (1996), 458-469.

[16] K. S. Park,R. Prasad, *Semi-slant submersions*, Bulletin of the Korean Mathematical Society, 50 (3) (2013), 951-962.

[17] R. Prasad, S. Kumar, *Conformal anti-invariant submersions from nearly Kähler Manifolds*, Palestine Journal of Mathematics, 8 (2) (2019), 234-247.

[18] R. Prasad, S. S. Shukla, S. Kumar, *On Quasi-bi-slant submersions*, Mediterranean Journal of Mathematics, 16 (6) (2019), 1-8.

[19] R. Prasad, P. K. Sigh, S. Kumar, *On quasi bi-slant submersions from Sasakian manifolds onto Riemannian manifolds*, Afrika Matematika, 32 (3) (2021), 403-417.

[20] R. Prasad, P. K. Singh, S. Kumar, *Conformal semi-slant submersions from Lorentzian para Kenmotsu manifolds*, Tbilisi Mathematical Journal, 14(1) (2021), 191-209.

[21] R. Prasad, M. A. Akyol, P. K. Singh, S. Kumar, *On Quasi bi-slant submersions from Kenmotsu manifolds onto any Riemannian manifolds*, Journal of Mathematical Extension, 16 (6) (2022), 1-25.

[22] R. Prasad, M. A. Akyol, S. Kumar, P. K. Singh, *Quasi bi-slant submersions in contact geometry*, Cubo (Temuco), 24 (1) (2022), 1-20.

[23] B. Şahin, *Riemannian submersions, Riemannian maps in Hermitian geometry, and their applications*, Elsiever, 2017.

[24] B. Şahin, *Anti-invariant Riemannian submersions from almost Hermitian manifolds*, Open Mathematics, 8 (3) (2010), 437-447.

[25] B. Şahin B., *Slant submersions from almost Hermitian manifods*, Bulletin Mathematique de la Societe des Sciences Mathematiques de Roumanie, 54 (1) (2011), 93-105.

[26] B. Şahin, *Semi-invariant submersions from almost Hermitian manifolds*, Canadian Mathematical Bulletin, 56 (1) (2013), 173-183.

[27] B. Şahin, *Circles along a Riemannian map and Clairaut Riemannian maps*, Bulletin of the Korean Mathematical Society, 54 (1) (2017), 253-264.

[28] H. M. Taştan, S. Gerdan, *Clairaut anti-invariant submersions from Sasakian and Kenmotsu manifolds*, Mediterranean Journal of Mathematics, 14 (6) (2017), 235-249.

[29] H. M. Taştan, S. Gerdan Aydin, *Clairaut anti-invariant submersions from cosymplectic manifolds*, Honam Mathematical Journal, 41 (4) (2019), 707-724.

[30] B. Watson, *Almost Hermitian submersions*, Journal of Differential Geometry, 11 (1) (1976), 147-165.

[31] A. Yadav, K. Meena, *Canti-invariant Riemannian maps from Kahler Manifolds*, Mediterranean Journal of Mathematics, 19 (3) (2022), 1-19.

**SUSHIL KUMAR**
DEPARTMENT OF MATHEMATICS
SHRI JAI NARAIN MISRA POST GRADUATE COLLEGE
LUCKNOW
INDIA
*E-mail address*: sushilmath20@gmail.com

**PUNIT KUMAR SINGH**
DEPARTMENT OF MATHEMATICS AND ASTRONOMY
UNIVERSITY OF LUCKNOW
LUCKNOW
INDIA
*E-mail address*: singhpunit1993@gmail.com

**RAJENDRA PRASAD**
DEPARTMENT OF MATHEMATICS AND ASTRONOMY
UNIVERSITY OF LUCKNOW
LUCKNOW
INDIA
*E-mail address*: rp.manpur@rediffmail.com