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ON EMBEDDING HILBERT ALGEBRAS IN BCK-ALGEBRAS

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Abstract. We prove that the class of Hilbert algebras maybe embedded into the class of all BCK-algebras.

The concept of Hilbert algebra was introduced in early 50-ties by L.Henkin and T.Skolem for some investigations of implication in intuicionistic and other non-classical logics. In 60-ties, these algebras were studied especially by A.Horn and A.Diego from algebraic point of view. A.Diego proved (cf. [5]) that Hilbert algebras form a variety which is locally finite. Hilbert algebras were treated by D.Busneag (cf. [2], [3]) and Y.B.Jun (cf. [10]) and recent of their filters forming deductive systems were recognized. Fuzzy deductive systems are described by S.M.Hong and Y.B.Jun (cf. [9] and [11]). I.Chajda and R.Halaš introduced in [4] the concept of ideals in Hilbert algebras and described connections between such ideals and congruences. In [7] is proved that every ideal is a deductive system.

BCK-algebras was introduced by K.Iséki in his paper [8]. The notion of BCK-algebras is inspired by BCK logic, i.e. an implicational logic based on *modus ponens* and the following axioms scheme (for details see for example [1]):

Axiom **B** $\quad \mathcal{A} \supset \mathcal{B}. \supset (\mathcal{C} \supset \mathcal{A}) \supset (\mathcal{C} \supset \mathcal{B})$
Axiom **C** $\quad \mathcal{A} \supset (\mathcal{B}. \supset \mathcal{C}). \supset .(\mathcal{A}) \supset \mathcal{C})$
Axiom **K** $\quad \mathcal{A} \supset (\mathcal{B} \supset \mathcal{A})$

In the similar way BCI-algebras, which are a generalization of BCK-algebras, are inspired by BCI logic, positive implicative BCK-algebras by positive implicative BCK-logic et cetera. In many cases, the connection between such algebras and corresponding logics is much stronger. In many cases one can give a translation procedure which translates all well formed formulas and all theorems of a given logic into terms and theorems of the corresponding algebra (cf. [1]). In some cases one can give also an inverse procedure.

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Nevertheless the study of algebras motivated by known logic is interesting and useful for corresponding logics, also in the case when an inverse translation procedure not exists. Such study (in general) gives a new inspiration to the investigation of corresponding logics.

Since there exist various modifications of the definition of Hilbert algebra, we use that of [2].

Definition 1. A *Hilbert algebra* is a triplet $\mathcal{H} = (H; *, \mathbf{1})$, where H is a nonempty set, $*$ is a binary operation and $\mathbf{1}$ is a fixed element of H such that the following axioms hold for each $x, y, z \in H$:

(I) $x * (y * x) = \mathbf{1}$,
(II) $(x * (y * z)) * ((x * y) * (x * z)) = \mathbf{1}$,
(III) $x * y = \mathbf{1}$ *and* $y * x = \mathbf{1}$ *imply* $x = y$.

The following result was proved (cf. for example [5]):

Lemma 1. *Let* $\mathcal{H} = (H; *, \mathbf{1})$ *be a Hilbert algebra and* $x, y, z \in H$. *Then*

(1) $x * x = \mathbf{1}$,
(2) $\mathbf{1} * x = x$,
(3) $x * \mathbf{1} = \mathbf{1}$,
(4) $x * (y * z) = y * (x * z)$,
(5) $x * (y * z) = (x * y) * (x * z)$. □

Lemma 2. *Let* $\mathcal{H} = (H; *, \mathbf{1})$ *be a Hilbert algebra and* $x, y, z \in H$. *Then*

(6) $x * ((x * y) * y) = \mathbf{1}$,
(7) $(x * y) * ((y * z) * (x * z)) = \mathbf{1}$,
(8) $(y * z) * ((x * y) * (x * z)) = \mathbf{1}$.

Proof. (6) is a consequence of (4) and (1). (7) is proved in [5]. (8) follows from (7) and (4). □

Definition 2. A nonempty set X with a constant 0 and a binary operation denoted by juxtaposition is called a *BCK-algebra* if for all $x, y, z \in X$ the following axioms hold:

(i) $((xy)(zy))(xz) = 0$,
(ii) $(x(xy))y = 0$,
(iii) $xx = 0$,
(iv) $0x = 0$,
(v) $x0 = x$,
(vi) $xy = 0$ and $yx = 0$ imply $x = y$.

A right distributive BCK-algebra, i.e. a BCK-algebra satisfying

$$(xy)z = (xz)(yz)$$

is called *positive implicative.*

From the above Lemmas follows

Theorem. *An algebra* $(G, *, \mathbf{1})$ *is a Hilbert algebra iff its dual algebra* $(G, \cdot, \mathbf{1})$, *where* $xy = y * x$ *is a positive implicative BCK-algebra.* □

Corollary 1. *Hilbert logics and positive implicative logics are anti-isomorphis, i.e. there exists translation procedure which translates well formed formulas and theorems of a Hilbert logic into terms and theorems of the corresponding implicative BCK logic.* □

Corollary 2. *The class of all Hilbert algebras maybe embedded into the class of all BCK-algebras.* □

Corollary 3. *A non-empty subset A of a Hilbert algebra* $(G, *, \mathbf{1})$ *is a deductive system iff it is an ideal of a dual BCK-algebra* $(G, \cdot, \mathbf{1})$. □

From results obtained in [7] follows also

Corollary 4. *The class of all positive implicative BCK-algebras is a variety in which every congruence is determined by its kernel.* □

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