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SOME CHARACTERIZATIONS OF LORENTZIAN SPHERICAL SPACE-LIKE CURVES

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Abstract. In this paper, some characterizations of a unit speed space-like curve whose image lies on a Lorentzian sphere in R_1^3 Minkowski 3-space are given.

1. Introduction

In Euclidean space R^3 a spherical unit speed curves and its characterizations are given in [3].

Instead of space R^3 let us consider the Minkowski 3-space R_1^3 provided with Lorentzian inner product

$$< \mathbf{a}, \mathbf{b} >= a_1b_1 + a_2b_2 - a_3b_3.$$

Here $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3) \in R^3$

Let (α) be a curve in space R_1^3, α' is the tangent vector for every $s \in$ $\in$ ICR. If

$$\begin{array}{ll} < \alpha', \alpha' >> 0, & \text{then } (\alpha) \text{ is a space-like curve,} \\ < \alpha', \alpha' >< 0, & \text{then } (\alpha) \text{ is time-like curve,} \\ < \alpha', \alpha' >= 0, & \text{then } (\alpha) \text{ is a light-like curve[2].} \end{array}$$

The Lorentzian sphere of radius 1 in R_1^3 is defined by

$$S_1^2 = \{\mathbf{a} = (a_1, a_2, a_3) \in R_1^3 | < \mathbf{a}, \mathbf{a} >= 1\}.$$

Let, $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ be tangent, principal normal and binormal unit vectors on $\alpha(s)$ space-like curve respectively. In $[\mathbf{t}, \mathbf{n}, \mathbf{b}]$ Frenet trihedron $\mathbf{b}$ is time-like unit vector, $\mathbf{t}$, $\mathbf{n}$ are space-like unit vectors. That is,

$$(1) \qquad \begin{aligned} < \mathbf{t}, \mathbf{t} > &= < \mathbf{n}, \mathbf{n} >= 1, \quad < \mathbf{b}, \mathbf{b} >= -1 \\ < \mathbf{t}, \mathbf{n} > &= < \mathbf{n}, \mathbf{b} >=< \mathbf{b}, \mathbf{t} >= 0 \\ \mathbf{t} \wedge \mathbf{n} &= \mathbf{b}, \quad \mathbf{n} \wedge \mathbf{b} = -\mathbf{t}, \quad \mathbf{b} \wedge \mathbf{t} = -\mathbf{n}. \end{aligned}$$

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Frenet derivative formulas are

$$\frac{dt}{ds} = \rho n, \quad \frac{dn}{ds} = -\rho t + \tau b, \quad \frac{db}{ds} = \tau n. \tag{2}$$

On the point of $\alpha(s)$ space-like curve, ρ and τ are called curvature and torsion respectively ([1], [4]).

2. Lorentzian Spherical Space-Like Curves

Theorem 1. *Let $\alpha(s)$ be a unit speed space-like curve whose image lies on a Lorentzian sphere of radius r and center $\mathbf{m}$ in R_1^3 Minkowski 3-space. Then $\rho \neq 0$. If $\tau \neq 0$ then*

$$\alpha - \mathbf{m} = -R\mathbf{n} + R'T\mathbf{b}.$$

Here $R = \frac{1}{\rho}$, $T = \frac{1}{\tau}$.

Proof. We have

$$< \alpha(s) - \mathbf{m}, \quad \alpha(s) - \mathbf{m} >= r^2,$$

so that

$$0 =< \alpha(s) - \mathbf{m}, \quad \alpha(s) - \mathbf{m} >' = 2 < \alpha(s) - \mathbf{m}, \mathbf{t} > . \tag{3}$$

Then

$$0 =< \alpha(s) - \mathbf{m}, \ \mathbf{t} >' =< \mathbf{t}, \ \mathbf{t} > + < \alpha(s) - \mathbf{m}, \ \mathbf{t}' > = 1 + < \alpha(s) - \mathbf{m}, \ \rho\mathbf{n} >,$$

$$\rho < \alpha(s) - \mathbf{m}, \ \mathbf{n} >= -1 \neq 0. \tag{4}$$

Thus $\rho \neq 0$.

Assume $\tau \neq 0$, $\alpha(s) - \mathbf{m} = a\mathbf{t} + b\mathbf{n} + c\mathbf{b}$ where the coefficients a, b, c may be found by (1), (3) and (4). Then

$$\begin{aligned} < \alpha(s) - \mathbf{m}, \ \mathbf{t} > &= a = 0, \\ < \alpha(s) - \mathbf{m}, \ \mathbf{n} > &= b = -\frac{1}{\rho} = -\mathbf{R}, \\ < \alpha(s) - \mathbf{m}, \ \mathbf{b} > &= -c. \end{aligned}$$

Since $< \alpha(s) - \mathbf{m}, \mathbf{n} >= -R$,

$$-R' =< \alpha(s) - \mathbf{m}, \ \mathbf{n} >' =< \alpha(s) - \mathbf{m}, \ -\rho\mathbf{t} + \tau\mathbf{b} >= \tau < \alpha(s) - \mathbf{m}, \ \mathbf{b} > .$$

Hence

$$-c =< \alpha(s) - \mathbf{m}, \ \mathbf{b} >= -R'T.$$

Thus $\alpha(s) - \mathbf{m} = -R\mathbf{n} + R'T\mathbf{b}$,

$$r^2 =< \alpha(s) - \mathbf{m}, \ \alpha(s) - \mathbf{m} >= R^2 - (R'T)^2. \quad \blacksquare$$

Theorem 2. *Let $\alpha(s)$ be a unit speed space-like curve with $R \neq 0$, $T \neq 0$ and $R = \frac{1}{\rho}$, $T = \frac{1}{\tau}$. Assume $R^2 - (R'T)^2 = r^2 = \text{constant}$, where $r > 0$. Then image of α lies on a Lorentzian sphere of radius r.*

Proof. We will show that the following equation is constant

$$\mathbf{m} = \alpha + R\mathbf{n} - R'T\mathbf{b}.$$

Then $\mathbf{m}$ should be the center of the Lorentzian sphere. We have

$$\mathbf{m}' = \mathbf{t} + R'\mathbf{n} + R'\mathbf{n} - (R'T)'\mathbf{b} - (R'T)\mathbf{b}',$$

using (2) we get,

$$\mathbf{m}' = \left(\frac{R}{T} - R''T - R'T'\right)\mathbf{b}.$$

Since

$$R^2 - (R'T)^2 = r^2,$$

we get $RR' - R'T(R''T + R'T') = 0$.

Hence

$$\frac{R}{T} - R''T = R'T'.$$

Then $\mathbf{m}' = \mathbf{0}$.

Let $\mathbf{m} = \mathbf{c}$. Then we get

$$\alpha - \mathbf{c} = -R\mathbf{n} + R'T\mathbf{b}.$$

This shows that α lies on a Lorentzian sphere with center c and radius r.

■

Theorem 3. *If $\alpha(s)$ is a space-like unit speed curve with $R \neq 0$, $T \neq 0$, then $\alpha(s)$ lies on a Lorentzian sphere if ond only if*

$$R\tau = \left(\frac{R'}{\tau}\right)'.$$

Proof. Let $\alpha(s)$ be a Lorentzian spherical unit speed curve. Then by Theorem 1

$$r^2 = R^2 - (R'T)^2,$$

where r is the radius of the Lorentzian sphere. If we differentiate, we obtain the following equation

$$\frac{R}{T} = R''T + R'T' = (R'T)'.$$

Then

$$R\tau = \left(\frac{R'}{\tau}\right)'.$$

We now assume that $R\tau = \left(\frac{R'}{\tau}\right)'$. Hence

$$\frac{R}{T} = R''T + R'T'.$$

Multiplying with $2R'T$, we get

$$2RR' - 2R'T(R''T + R'T') = 0,$$

whence the differential of

$$R^2 - (R'T)^2 = r^2 = \text{constant}.$$

According to the Theorem 2 $\alpha(s)$ lies on a Lorentzian sphere.

Theorem 4. *A unit speed space-like curve $\alpha(s)$ lies on a Lorentzian sphere if and only if $\rho > 0$ and there exists a differentiable function $f(s)$ with $f\tau = R'$, $f' - R\tau = 0$.*

Proof. Let $\alpha(s)$ be a unit speed space-like curve lies on a Lorentzian sphere. From Theorem 3 we get

$$R\tau = \left(\frac{R'}{\tau}\right)'.$$

If we have

$$f(s) = \frac{R'}{\tau}, \qquad f' = R\tau$$

is obtained. That is $f\tau = R'$ and $f' - R\tau = 0$.

Conversely, let $f\tau = R'$, $f' - R\tau = 0$.

Then $f = \frac{R'}{\tau}$, $f' = R\tau$. We can write $R\tau = \left(\frac{R'}{\tau}\right)'$. From Theorem 3 $\alpha(s)$ space-lice curve lies on a Lorentzian sphere. ■.

Theorem 5. *Unit speed space-like curve $\alpha(s)$ lies on a Lorentzian sphere if and only if there are constants A and B with*

$$\rho\left(Ach\left(\int_0^s \tau ds\right) - Bsh\left(\int_0^s \tau ds\right)\right) \equiv 1.$$

Proof. Let $\frac{1}{\rho} = Ach\int_0^s \tau ds - Bsh\int_0^s \tau ds$. From differentiation we write

$$\frac{d}{ds}\left(\frac{1}{\rho}\right) = \tau\left[Ash\int_0^s \tau ds - Bch\int_0^s \tau ds\right].$$

If the function f is defined by

$$f(s) = Ash\int_0^s \tau ds - Bch\int_0^s \tau ds,$$

then it satisfies the conditions

$$f\tau = R', \qquad f' = R\tau.$$

From Theorem 4 the proof of the sufficiency is complete.

Suppose that $\alpha(s)$ is a unit speed space-like curve lies on a Lorentzian sphere. Let us define c^2-function θ and c^1-functions $g(s)$ and $h(s)$ on $[0, L]$ by

$$\theta(s) = \int_0^s \tau ds, \quad g(s) = \frac{1}{\rho} ch\theta - f(s) sh\theta, \quad h(s) = \frac{1}{\rho} sh\theta - f(s) ch\theta.$$

From differentatiation with respect to s and taking account of $\theta(s) = \int_0^s \tau ds$ and $f\tau = R'$, $f' = R\tau$ we find that g' and h' are both identicaly zero. Therefore $g(s) = A$, $h(s) = B$, where A and B are constants. We write

$$A = \frac{1}{\rho} ch\theta - f(s) sh\theta, \qquad B = \frac{1}{\rho} sh\theta - f(s) ch\theta.$$

Solving the resulting equations for $\frac{1}{\rho}$, we get

$$\frac{1}{\rho} = Ach\theta - Bsh\theta,$$

that is,

$$\rho \left(Ach \left(\int_0^s \tau ds \right) - Bsh \left(\int_0^s \tau ds \right) \right) \equiv 1 \quad \blacksquare$$

3. References

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