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A FIXED POINT THEOREM FOR MAPPINGS IN d-COMPLETE TOPOLOGICAL SPACES

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Abstract. A general fixed point for four mappings satisfying an implicit relation in d-complete topological spaces which generalize Theorem 3.7 of [1] is proved.

1. Introduction

Let (X, τ) be a topological space and $d : X \times X \to [0, \infty)$ such that $d(x, y) = 0$ if and only if $x = y$. X is said to be d-complete [3] if

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$$

implies that the sequence $\{x_n\}$ is convergent in (X, τ). Complete metric spaces and complete quasi-metric spaces are examples of d-complete topological spaces.

Recently, Hicks [3], Hicks and Rhoades [4], Saliga [5] proved several fixed point theorems in d-complete topological spaces.

Let $T : X \to X$ be a mapping. T is ω-continuous at $x \in X$ if $x_n \to x$ implies $Tx_n \to Tx$ as $n \to \infty$.

Recently, Cho, Sharma and Zahu [1] introduced the notion of semi-compatibility in topological spaces.

Definition [1]. *Let S and T be mappings from a topological spaces (X, τ) into itself. The mappings S and T are said to be semi-compatible if they hold the following conditions:*

$$D_1 : Sp = Tp \quad \text{for some } p \in X \text{ implies } STp = TSp;$$

D_2: *The ω-continuity of T at a point p in X implies* $\lim STx_n = Tp$, *whenever $\{x_n\}$ a sequence in X such that* $\lim Sx_n = \lim Tx_n = p$ *for some p in X.*

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2. Implicit relations

Let D_4 be the set of all real continuous functions $F(t_1, \dots, t_4)$ satisfying the following conditions:

D_h: there exists $h \in [0,1)$ such that for every $u \geq 0$, $v \geq 0$ with

$$\text{a) } F(u,v,v,u) \leq 0, \quad \text{or} \quad \text{b) } F(u,v,u,v) \leq 0$$

we have $u \leq hv$.

$$D_u : F(u,u,0,0) > 0, \quad \forall u > 0.$$

Ex. 1. $F(t_1, \dots, t_4) = t_1 - \max\{t_2, t_3, t_4\}$ where $m \in [0,1)$.

D_h: Let $u > 0$, $v \geq 0$ and $F(u,v,v,u) = u - m\max\{u,v\} \leq 0$. If $u \geq v$ then $u(1-m) \leq 0$ is a contradiction. Thus $u < v$ and $u \leq hv$, where $h = m < 1$. Similarly, $u > 0$, $v \geq 0$ and $F(u,v,u,v) \leq 0$ implies $u \leq hv$. If $u = 0$ then $u \leq hv$.

$$D_u : F(u,u,0,0) = u(1-m) > 0, \quad \forall u > 0.$$

Ex. 2 $F(t_1, \dots, t_4) = t_1 - (at_2^k + bt_3^k + ct_4^k)^{1/k}$ where $k > 0$; $a,b,c \geq 0$ and $a+b+c < 1$.

D_h: If $F(u,v,v,u) \leq 0$ then $u^k - av^k - bv^k - cu^k \leq 0$ which implies $a \leq h_1 v$ where $h_1 = \left(\frac{a+b}{1-c}\right)^{1/2} \in [0,1)$. Similarly, $F(u,v,u,v) \leq 0$ implies $u \leq h_2 v$ where $h_2 = \left(\frac{a+c}{1-b}\right)^{1/k} \in [0,1)$. Thus $u \leq hv$ where $h = \max\{h_1, h_2\}$.

$$D_u : F(u,u,0,0) = u(1-a^{1k}) > 0, \quad \forall u > 0.$$

Ex. 3. $F(t_1, \dots, t_4) = t_1 - (at_2^2 + bt_3^2 + ct_4^2 + dt_1t_4)^{1/2}$ where $a,b,c,d \geq 0$ and $a+b+c+d < 1$.

D_h: Let $u > 0$, $v \geq 0$ and $F(u,v,v,u) = u - (av^2 + bv^2 + cu^2 + du^2)^{1/2} \leq 0$. If $u \geq v$ then $u(1 - \sqrt{a+b+c+d}) \leq 0$, is a contradiction. Then $u < v$ and $u \leq hv$ where $h = \sqrt{a+b+c+d} < 1$. Similarly, $u > 0$, $v \geq 0$ and $F(u,v,u,v) < 0$ implies $u \leq hv$. If $u = 0$ then $u \leq hv$.

Ex. 4. $F(t_1, \dots, t_4) = t_1 \max\{t_1, t_3, t_4\} - a\min\{t_2, t_3, t_4\}$ where $0 \leq a < 1$.

D_h: Let $u > 0$, $v \geq 0$ and $F(u,v,v,u) = u\max\{u,v\} - av\min\{u,v\} \leq 0$. If $u \geq v$ then $u^2(1-a) \leq 0$, is a contradiction. Thus $u < v$ and $u \leq hv$ where $h = \sqrt{a}$. Similarly, $F(u,v,u,v) \leq 0$ implies $u \leq hv$. If $u = 0$ then $u \leq hv$.

$$D_u : F(u,u,0,0) = u^2(1-a) > 0, \quad \forall u > 0.$$

In [2] Delbosco consider the family P of all real-valued function $p : R_+^3 \to [0,\infty)$ satisfying the following conditions:

(1) p is continuous in R_+^3,

(2) $p(1,1,1) = h < 1$, where $h \in [0,1)$,

(3) if $u, v \geq 0$ and

$$(a')\ u \leq p(v,v,u) \quad \text{or} \quad (b')u \leq p(v,u,v) \quad \text{or} \quad (c')u \leq p(u,v,v)$$

then $u \leq hv$.

Remark. If $F(t_1, \dots, t_4) = t_1 - p(t_2, t_3, t_4)$ then the conditions (a) and (b) from D_4 are satisfied. The conditions (c') implies condition D_u. Indeed, if $u > 0$ and $F(u,u,0,0) = u - p(u,0,0) \leq 0$ then $u \leq p(u,0,0)$ implies $u \leq \leq h_0 = 0$. This is a contradiction. Delbosco has proved the following theorem.

Theorem 2.1 *Let T be a self-mapping of a complete metric space (X,d). If*

$$d(Tx,Ty) \leq p(d(x,y), d(x,Tx), d(y,Ty))$$

for all $x, y \in X$, where $p \in P$, then T has a unique fixed point.

A generalization of Theorem 2.1 is proved in [1].

Theorem 2.2 [1]. *Let A, B, S, T be mappings from a Hausdorff d-complete topological space (X, τ) into itself satisfying the conditions:*

(2.1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$.

(2.2) the pairs A, S and B, T are semi-compatible mappings,

(2.3) one of A, B, S and T is ω-continuous,

(2.4) $d(Ax, By) \leq p((d(Sx,Ty), d(Sx,Ax), d(Ty,By))$,

for all x, y in X where $p \in P$. Then A, B, S and T have a unique common fixed point.

The purpose of this paper is to prove a generalization of Theorem 2.2.

3. Main results

Theorem 3.1. *Let (X,τ) be a d-topological space and A, B, S, T: $: (X,\tau) \to (X,\tau)$ satisfying the inequality*

$$(1) \qquad F\big(d(Ax,By), d(Sx,Ty), d(Sx,Ax), d(Ty,By)\big) \leq 0$$

for all x, y in X, where F satisfies property (D_u). Then A, B, S, T have at most one common fixed point.

Proof. Suppose that A, B, S, T have two common fixed points z and z', with $z \neq z'$. By (1) we have succesively

$$F\big(d(Az,Bz'), d(Sz,Tz'), d(Sz,Az), d(Tz',Bz')\big) \leq 0)$$
$$F\big(d(z,z'), d(z,z'), 0, 0\big) \leq 0) \quad \text{is a contradiction of } (D_u).$$

Theorem 3. *Let A, B, S and T be mappings from a Hausdorff d-complete topological space (X,τ) into itself satisfying the conditions (2.1), (2.2), (2,3) and (1) for all x, y in X, where $F \in D_4$. Then A, B, S, T have a unique common fixed point.*

Proof. By (2.1), since $A(X) \subset T(X)$, for any arbitrary point $x_\circ$ in X there exists a point x_1 in X such that $Ax_\circ = Tx_1$. Since $B(X) \subset S(X)$, for this point x_1, we can choose a point x_2 in X such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$(2) \qquad Tx_{2n+1} = Ax_{2n} = y_{2n} \quad \text{and} \quad Sx_{2n+2} = Bx_{2n+1} = y_{2n+1},$$

for all $n = 0, 1, 2, \ldots$. Letting $d_n = d(y_n, y_{n+1})$ and applying (1) we have succesively

$$F\Big(d\left(Ax_{2n+2}, Bx_{2n+1}\right), d\left(Sx_{2n+2}, Tx_{2n+1}\right), d\left(Sx_{2n+2}, Ax_{2n+2}\right),$$
$$d\left(Tx_{2n+1}, Bx_{2n+1}\right)\Big) \leq 0$$

which by (b) implies:

$$d_{2n+1} \leq hd_{2n}.$$

Similarly, we have succesively

$$F\Big(d\left(Ax_{2n}, Bx_{2n+1}\right), d\left(Sx_{2n}, Tx_{2n+1}\right), d\left(Sx_{2n}, Ax_{2n}\right),$$
$$d\left(Tx_{2n+1}, Bx_{2n+1}\right)\Big) \leq 0,$$
$$F\left(d_{2n}, d_{2n-1}, d_{2n-1}, d_{2n}\right) \leq 0,$$

which by (a) implies:

$$d_{2n} \leq hd_{2n-1}.$$

An induction givens

$$d_n \leq h^{n-1} d_\circ$$

and thus $\sum_{n=1}^{\infty} d_n < \infty$. It follows that $\sum_{n=1}^{\infty} d(y_n, y_{n+1})$ is convergent. Since X is d-complete, $\{y_n\}$ converges to some point z in X and hence the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to the point z.

Now, suppose that T is ω-continuous. Since B and T are semicompatible and the subsequneces $\{Bx_{2n+1}\}$, $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to the point z, by the property (D_2) we have

$$BTx_{2n+1}, \quad TTx_{2n+1} \to Tz \quad \text{as} \quad n \to \infty.$$

Putting $x = x_{2n}$ and $y = Tx_{2n+1}$ in (1) we have

$$F\Big(d\left(Ax_{2n}, BTx_{2n+1}\right), d\left(Sx_{2n}, TTx_{2n+1}\right), d\left(Sx_{2n}, Ax_{2n}\right), d\left(TTx_{2n+1}, BTx_{2n+1}\right)\Big) \leq 0. \tag{3}$$

Letting $n \to \infty$ in (3), we have

$$F(d(z, Tz), d(z, Tz), 0, 0) \leq 0.$$

This is a contradiction of (D_u) if $d(z, Tz) > 0$ and so $Tz = z$. Again replacing z by x_{2n} and y by z in (1), we have

$$F\Big(d(x_{2n}, Bz), d(Sx_{2n}, Tz), d(Sx_{2n}, Ax_{2n}), d(z, Bz)\Big) \leq 0. \tag{4}$$

As $n \to \infty$ in (4), we have

$$F\Big(d(z, Bz), 0, 0, d(z, Bz)\Big) \leq 0$$

which by (a) implies that $d(z, Bz) \leq h_0$. Thus $z = Bz$. Since $B(X) \subset S(X)$, there exists a point u in X such that $Bz = Su = z$. By (1) we have

$$F\Big(d(Au, Bz), d(Su, Tz), d(Su, Au), d(Tz, Bz)\Big) \leq 0,$$

$$F\Big(d(Au, z), 0, d(Au, z), 0)\Big) \leq 0,$$

which by (b) implies that $d(Au, z) \leq h_0$. Thus $Au = z$. But since A and S are semi-compatible and $Au = Su = z$, by the property D_1 we have $Az = ASu = SAu = z$. By using (1) we have sucesively

$$F\Big(d(Az, Bz), d(Sz, Tz), d(Sz, Az), d(Tz, Bz)\Big) \leq 0.$$

Then from $F(d(Az, z), d(Az, z), 0, 0) \leq 0$, follows the contradiction of (D_u) if $d(Az, z) > 0$. Thus $Az = z$. Therefore $Az = Bz = Sz = Tz = z$, that z is a common fixed point of A, B, S and T. The uniqueness of the common fixed point z follows from Theorem 3.1. Similarly, we can prove the preceding facts, when A or B or S is ω-continuous.

4. References

[1] Y. J. Cho, B. K. Sharma and D. R. Sahy: *Semi-compatibility and Fixed points*, Math. Japonica, **42** (1995), 91-98.

[2] D. Delbosco: *A unified approach for all contractive mappings*, Jnanbha, **16** (1986), 1-11.

[3] T. L. Hicks: *Fixed point theorems for d-complete topological spaces I*, Internat. J. Math. and Math.Sci., **15** (1992), 435-440.

[4] T. L. Hicks and B. E. Rhoades: *Fixed point theorems for d-complete topological spaces II*, Math. Japonica, **37** (1992), 847-853.

[5] L. M. Saliga: *Fixed point theorems for non-self maps in d-complete topological spaces*, Internat.J.Math. and Math.Sci., **19** (1996), 103-110.

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