MATEMATICA MORAVICA
VOL. 3 (1999), 49–52

A NOTE ON GENERALIZED INVERSE FUNCTIONS

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Abstract. Robin Harte generalized an observation of Hochwald and Morell. In this note we offer a generalization of Harte's result.

Let $\mathcal{A}$ be an associative ring with identity 1. We shall write $\mathcal{A}^{-1}$ for the invertible group. The *commutant* of a subset $\mathcal{K} \subset \mathcal{A}$ is the set

$$\text{comm}_{\mathcal{A}}(\mathcal{K}) = \{a \in \mathcal{A} : ab = ba \quad \text{for each} \quad b \in \mathcal{K}\}, \tag{0.1}$$

and in particular, the *centre* of $\mathcal{A}$ is given by

$$\text{Centre}\,(\mathcal{A}) = \text{comm}_{\mathcal{A}}(\mathcal{A}). \tag{0.2}$$

Following Harte [5] we shall call the ring $\mathcal{A}$ *centre-commutative* if

$$\text{for each} \quad a \in \mathcal{A} \quad \text{there is} \quad \lambda \in \text{Centre}\,(\mathcal{A}) \quad \text{for which} \quad a - \lambda \in \mathcal{A}^{-1}. \tag{0.3}$$

We write

$$\widehat{\mathcal{A}} = \{a \in \mathcal{A} : a \in a\mathcal{A}a\} \tag{0.4}$$

for the subset of those elements of $\mathcal{A}$ which have *generalized inverses*, and set

$$\mathcal{A}^{\mu} = \{a \in \mathcal{A} : a \in a^m\mathcal{A}a \quad \text{for some non-negative integer } m\}. \tag{0.5}$$

Let us recall that an element $a \in \mathcal{A}$ is said to have a *Drazin* inverse, or that a is *Drazin invertible* [4] if there exists $x \in \mathcal{A}$ such that

$$a^m = a^{m+1}x \qquad \text{for some non-negative integer } m, \tag{0.6}$$

$$x = ax^2, \quad \text{and} \quad ax = xa. \tag{0.7}$$

If a has Drazin inverse, then the smallest non-negative integer m in (0.6) above is called the *index* $i(a)$ of a. It is well known that there is at most one x such that equations (0.6) and (0.7) hold. The unique x is denoted by a^D and called the *Drazin* inverse of a. We write $\mathcal{A}^D$ for the subset of those elements of $\mathcal{A}$ which have Drazin inverses. Recall that if a has Drazin inverse, then a^D

AMS (MOS) Subject Classification 1991. Primary: 46H05, 39B70.

Key words and phrases: Generalized inverses, Drazin inverse (invertible), Harte's observation, Observation of Hochwald and Morell, Generalized Drazin inverse.

also has Drazin inverse, $i(a^D) \leq 1$, $(a^D)^D = a^2a^D$ and $((a^D)^D)^D = a^D$ [4]. If a has Drazin inverse then a may always be written as

$$a = a^{(c)} + a^{(n)}, \tag{0.8}$$

where $a^{(c)}, a^{(n)} \in \mathcal{A}$, $a^{(c)}$ has Drazin inverse, $i(a^{(c)}) \leq 1, a^{(c)}a^{(n)} = a^{(n)}a^{(c)} = 0$, and $(a^{(n)})^{i(a)} = 0$. The elements $a^{(c)}, a^{(n)}$ are unique. $a^{(c)}$ is called the *core* of a, and $a^{(n)}n$ the *nilpotent* part of a. Let us mention that in this case

$$a^{(c)} = a^2a^D \qquad \text{and} \qquad a^{(n)} = a - a^2a^D. \tag{0.9}$$

We shall refer to $a^{(c)} + a^{(n)}$ as the *core nilpotent* decomposition of a (see [1, 4,] or [7, 10] for recent applications).

Let us recall that Harte [5, Theorem 1] (see also [9, Teorema 5.8.1]) generalized an observation of Hochwald and Morell [6], which was stimulated by a question of Allan [1]. In this note we offer the following generalization of Harte's result.

Theorem 1. *Let $\mathcal{A}$ be a centre-commutative ring and $\mathcal{D} \subset \mathcal{A}$ a semigroup for which $\mathcal{A}^{-1} \subset \mathcal{D} \subset \mathcal{A}^{\mu}$. If $g : \mathcal{D} \mapsto \mathcal{A}$ is a mapping which satisfies, for each $a \in \mathcal{D}$,*

$$a^m = a^m g(a)a \tag{1.1}$$

for some non-negative integer $m = m(a)$, and

$$b \in \mathcal{A}^{-1} \cap \text{comm}_{\mathcal{A}}(a) \Longrightarrow g(a)g(b) = g(b)g(a), \tag{1.2}$$

then

$$a \in \mathcal{D} \Longrightarrow a \in a^m\mathcal{A}^{-1}a. \tag{1.3}$$

Proof. We claim that $\mathcal{D} \subset \mathcal{A}^D$. Let $a \in \mathcal{D}$. By Harte's proof [5, Theorem 1] we have

$$ag(a) = g(a)a. \tag{1.4}$$

Set $v = a^m g(a)^{m+1}$. Then obviously $av = va$. Now, by (1.1) we remark that $a^{m+k}g(a)^k = a^m$ for all $k = 1, 2, \ldots$. Hence by (1.4) we get

$$a^{m+1}v = a^{2m+1}[g(a)]^{m+1} = a^m. \tag{1.5}$$

Further

$$vav = a^{2m+1}g(a)^{2m+2} = a^{m+(m+1)}g(a)^{m+1}g(a)^{m+1} = a^m g(a)^{m+1} = v. \tag{1.6}$$

Thus a is Drazin invertible and $v = a^D$. Let us remark that $1 + a^{(n)}$ is inverible, where $a^{(n)} = a(1 - aa^D)$ is the nilpotent part of a. Now, it is easy to check that $u = a^D + (1 - aa^D)$ is invertible and that $u^{-1} = [a + (1 - aa^D)](1 + a^{(n)})^{-1}$. Finally,

$$a^m ua = a^m[a^D + (1 - aa^D)]a = a^m a^D a + a^m(1 - aa^D)a = a^m. \qquad \blacksquare$$

Corollary 2. (Harte [5, Theorem 1]). *If $\mathcal{A}$ is a centre-commutative ring, if $\mathcal{D} \subset \mathcal{A}$ is a semigroup for which $\mathcal{A}^{-1} \subset \mathcal{D} \subset \widehat{\mathcal{A}}$, and if $g : \mathcal{D} \mapsto \mathcal{A}$ is a mapping which satisfies, for each $a \in \mathcal{D}$,*

$$a = ag(a)a \tag{2.1}$$

and

$$b \in \mathcal{A}^{-1} \cap \text{comm}_{\mathcal{A}}(a) \Longrightarrow g(a)g(b) = g(b)g(a), \tag{2.2}$$

then

$$a \in \mathcal{D} \Longrightarrow a \in a\mathcal{A}^{-1}a. \tag{2.3}$$

Proof. Apply Theorem 1 for $m = 1$. ■

Remark. Recently Koliha [7] (see [8] for applications also) has introduced and investigated a generalized inverse (he called it a *generalized Drazin inverse*) in associative rings and Banach algebras, i.e., if $\mathcal{A}$ is a complex unital Banach algebra, then an element $a \in \mathcal{A}$ is said to have a *generalized Drazin* inverse if there exists $x \in \mathcal{A}$ such that

$$a - a^2x \quad \text{is quasinilpotent,} \tag{3.1}$$

$$x = ax^2 \quad \text{and} \quad ax = xa. \tag{3.2}$$

If a has generalized Drazin inverse, then there is at most one x such that equations (3.1) and (3.2) hold.

Concerning the proof of Theorem 1, we would like to finish this note with the following question:

Question. Let $\mathcal{A}$ be a complex unital Banach algebra and $a \in \mathcal{A}$. If there exists $x \in \mathcal{A}$ such that $a - a^2x$ is quasinilpotent and $ax = xa$ must a has generalized Drazin inverse?

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Received January 7, 1999.