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A REMARK ON THE LOCATION OF THE ZEROS OF POLYNOMIALS

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Abstract. In this paper we determine, in the complex plane, regions containing the zeros of the polynomial

$$(1) \qquad P(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n, \quad n \geq 3.$$

We also obtain two expressions which represent upper bounds for the moduli of the zeros of $P(z)$ with greater precision than those obtained by Cauchy and P. Montel.

1. The location of the zeros of the polynomial (1) in the complex plane, depending on its coefficients a_k, $k = 1, 2, \cdots, n$, was investigated by many authors (see e.g. [1] and [2]). Here we quote a result due to Cauchy [1, p.123] and a result due to P. Montel [3] which are respectively as follows:

(R_1): *All the zeros of the polynomial* (1) *lie in the region*

$$(2) \qquad |z| < 1 + A,$$

where

$$(3) \qquad A = \max |a_k|, \qquad k = 1, 2, \cdots, n.$$

(R_2): *All the zeros of the polynomial* (1) *lie in the region*

$$(4) \qquad |z| < 2M,$$

where

$$(5) \qquad M = \max |a_k|^{\frac{1}{k}}, \qquad k = 1, 2, \cdots, n.$$

2. In this paper, for the polynomial (1), the following theorem is proved.

Theorem. *For fixed positive parameter s,let*

$$(6) \qquad M = \max |a_k|^{\frac{1}{s+k-1}}, \qquad k = 1, 2, \cdots, n,$$

$$(7) \qquad M_1 = \max |a_j|^{\frac{1}{s+j-1}}, \qquad j = 1, 3, \cdots, u,$$

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(8) $$M_2 = \max |a_m|^{\frac{1}{s+m-1}}, \qquad m = 2, 4, \cdots, v,$$

where $u = n$, $v = n-1$, *when* n *is odd, respectively,* $u = n-1$, $v = n$, *when* n *is even. Then all the zeros of the polynomial* (1) *lie in the region*

(9) $$|z| < \frac{M_1^s + \sqrt{M_1^{2s} + 4M^2 + 4M_2^{s+1}}}{2}.$$

Proof. From (7) and (8) we have

(10) $$|a_1| \le M_1^s, \ |a_3| \le M_1^{s+2}, \cdots, |a_u| \le M_1^{s+u-1},$$

(11) $$|a_2| \le M_2^{s+1}, \ |a_4| \le M_2^{s+3}, \cdots, |a_v| \le M_2^{s+v-1}.$$

On account of $M = \max(M_1, M_2)$, we have

(12) $$M_1 \le M \quad \text{and} \quad M_2 \le M.$$

Taking into account (10), (11) and (12), from (1) for $|z| > M$ we have

$$|P(z)| \ge |z|^n \Big\{ 1 - |z|^{s-1} \Big(\Big[\frac{|a_1|}{|z|^s} + \frac{|a_3|}{|z|^{s+2}} + \cdots + \frac{|a_u|}{|z|^{s+u-1}} \Big] +$$

$$+ \Big[\frac{|a_2|}{|z|^{s+1}} + \frac{|a_4|}{|z|^{s+3}} + \cdots + \frac{|a_v|}{|z|^{s+v-1}} \Big] \Big) \Big\} >$$

$$> |z|^n \Big\{ 1 - |z|^{s-1} \Big(\Big[\Big(\frac{M_1}{|z|} \Big)^s + \Big(\frac{M_1}{|z|} \Big)^{s+2} + \cdots \Big] + \Big[\Big(\frac{M_2}{|z|} \Big)^{s+1} + \Big(\frac{M_2}{|z|} \Big)^{s+3} + \cdots \Big] \Big) \Big\} =$$

$$= |z|^n \Big\{ 1 - \frac{M_1^s |z|}{|z|^2 - M_1^2} - \frac{M_2^{s+1}}{|z|^2 - M_2^2} \Big\} \ge |z|^n \Big\{ 1 - \frac{M_1^s |z| + M_2^{s+1}}{|z|^2 - M^2} \Big\},$$

that is

(13) $$|P(z)| > |z|^n \Big\{ 1 - \frac{M_1^s |z| + M_2^{s+1}}{|z|^2 - M^2} \Big\}.$$

From (13) we have $|P(z)| > 0$ for $|z| \ge \frac{M_1^s + \sqrt{M_1^{2s} + 4M^2 + 4M_2^{s+1}}}{2}$. This means that $|P(z)| \ne 0$ at the points of the complex plane not contained in the region (9). From this we deduce that all the zeros of the polynomial (1) must be in the region (9), thus completing the proof of this Theorem.

3. Taking for parameter s different positive values we can obtain from the Theorem several particular results. Here we list two particular cases.

3.1. For $s = 1$ from the Theorem we obtain the following result:

(R_{31}): *All the zeros of the polynomial* (1) *lie in the region*

(14) $$|z| < \frac{M_1 + \sqrt{M_1^2 + 4M^2 + 4M_2^2}}{2},$$

where

(15) $$M = \max |a_k|^{\frac{1}{k}}, \qquad k = 1, 2, \cdots, n,$$

(16) $$M_1 = \max |a_j|^{\frac{1}{j}}, \qquad j = 1, 3, \cdots, u,$$

(17) $$M_2 = \max |a_m|^{\frac{1}{m}}, \qquad m = 2, 4, \cdots, v,$$

and where $u = n$, $v = n - 1$, *when* n *is odd, respectively,* $u = n - 1$, $v = n$, *when* n *is even.*

In view of (15), (16) and (17), we have

(18) $$M_1 \leq M \quad \text{and} \quad M_2 \leq M.$$

Keeping in mind (18), we see that

(19) $$\frac{M_1 + \sqrt{M_1^2 + 4M^2 + 4M_2^2}}{2} \leq 2M.$$

Taking into account (19) we conclude that the region (14) is contained in the Montel's region (4),except when $M_1 = M_2 = M$, in what case the two regions coincide.

3.2. The case when $s \to \infty$. From (6), (7) and (8) we have

(20) $$\lim_{s\to\infty} M = \lim_{s\to\infty} M_1 = \lim_{s\to\infty} M_2 = 1,$$

(21) $$\lim_{s\to\infty} M_1^s = A_1 = \max |a_j|, \qquad j = 1, 3, \cdots, u,$$

(22) $$\lim_{s\to\infty} M_2^{s+1} = A_2 = \max |a_m|, \qquad m = 2, 4, \cdots, v,$$

where $u = n$, $v = n - 1$, when n is odd, respectively, $u = n - 1$, $v = n$, when n is even.

Hawing in mind (20), (21) and (22), from the Theorem we obtain the following result:

(R_{32}): *All the zeros of the polynomial* (1) *lie in the region*

(23) $$|z| < \frac{A_1 + \sqrt{A_1^2 + 4 + 4A_2}}{2},$$

where

(24) $$A_1 = \max |a_j|, \qquad j = 1, 3, \cdots, u,$$

(25) $$A_2 = \max |a_m|, \qquad m = 2, 4, \cdots, v,$$

(26) $$A = \max |a_k|, \qquad k = 1, 2, \cdots, n,$$

and where $u = n$, $v = n - 1$, *when* n *is odd, respectively,* $u = n - 1$, $v = n$, *when* n *is even.*

In view of (24), (25) and (26), we have

(27) $$A_1 \leq A \quad \text{and} \quad A_2 \leq A.$$

Having in mind (27), we see that

$$\frac{A_1 + \sqrt{A_1^2 + 4 + 4A_2}}{2} \leq 1 + A. \tag{28}$$

Taking into account (28) we conclude that the region (23) is contained in the Cauchy's region (2), except when $A_1 = A_2 = A$ in what case the two regions coincide.

4. References

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