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A PROCEDURE FOR OBTAINING ITERATIVE FORMULAS OF HIGHER ORDER

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Abstract. In this paper a procedure for obtaining iterative formulas of higher order is obtained. In particular, a family of iterative formulas of higher order is given. The family includes several already known results.

1. Introduction

Let

$$x_{n+1} = f(x_n), \; n = 0, 1, 2, \cdots \tag{1}$$

be an iterative method for finding the root $x = \alpha$ of the real or complex equation $F(x) = 0$.

For the iterative method (1) which converges to $x = \alpha$, we say it is of order k if

$$|x_{n+1} - \alpha| = O\left(|x_n - \alpha|^k\right), \; n \to \infty. \tag{2}$$

If the function $f(x)$ is k times differentiable in a neighbourhood of the limit point $x = \alpha$, then the iterative method (1) is of order k if and only if

$$f(\alpha) = \alpha, \;\; f'(\alpha) = f''(\alpha) = \cdots = f^{(k-1)}(\alpha) = 0, \;\; f^{(k)}(\alpha) \neq 0. \tag{3}$$

This paper deals with a general procedure for obtaining iterative formulas of higher order.

2. A Theorem for Iterative Formulas of Higher Order

Starting from an iterative method of order k for finding the root $x = \alpha$ of the real or complex equation $F(x) = 0$, we give, in this paper, a procedure for obtaining iterative methods of order $\geq k + 1$. In this connection the following theorem is proved here.

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Theorem 1. *Let (1) be an iterative method of order $k \geq 2$. Let the function $f(x)$ be $k+1$ times differentiable in a neighbourhood of the limit point $x = \alpha$. Then for the function $g(x)$ of the form*

$$g(x) = G\big(f'(x)\big) \tag{4}$$

k times differentiable in the neighbourhood of the limit point $x = \alpha$ such that

$$g(\alpha) = 0, \tag{5}$$

$$g'(x) = \frac{1}{k} h(x) f''(x) \tag{6}$$

and

$$h(\alpha) = 1, \tag{7}$$

formula

$$x_{n+1} = f(x_n) - g(x_n)\big(x_n - f(x_n)\big), \quad n = 0, 1, 2, \cdots \tag{8}$$

is an iterative method of order $\geq k+1$.

Proof. In the method (1) the iteration function is $f(x)$, and in the method (8) the iteration function is

$$\varphi(x) = f(x) - g(x)(x - f(x)). \tag{9}$$

For the function $\varphi(x)$ we shall prove that

$$\varphi(\alpha) = \alpha, \quad \varphi'(\alpha) = \varphi''(\alpha) = \cdots = \varphi^{(k)}(\alpha) = 0. \tag{10}$$

By hypothesis, (1) is an iterative method of order $k \geq 2$ and therefore the relations (3) hold.

From (9) and (6) we have, respectively

$$\begin{aligned} \varphi^{(r)}(x) &= f^{(r)}(x) - g^{(r)}(x)\big(x - f(x)\big) - r g^{(r-1)}(x)\big(1 - f'(x)\big) + \\ &+ \binom{r}{2} g^{(r-2)}(x) f''(x) + \cdots + r g'(x) f^{(r-1)}(x) + g(x) f^{(r)}(x) \end{aligned} \tag{11}$$

and

(12)

$$\begin{aligned} g^{(r-1)}(x) = \frac{1}{k} \Bigg(& h^{(r-2)}(x) f''(x) + (r-2) h^{(r-3)}(x) f'''(x) + \\ & + \binom{r-2}{2} h^{(r-4)}(x) f^{(4)}(x) + \cdots + (r-2) h'(x) f^{(r-1)}(x) + h(x) f^{(r)}(x) \Bigg). \end{aligned}$$

For $k \geq 2$, in view of (3), we obtain from (9)

$$\varphi(\alpha) = \alpha. \tag{13}$$

Because of (3) and (5), for $r = 1$, we obtain from (11)

(14) $$\varphi'(\alpha) = 0.$$

For $k = 2$, in view of (3) and (7), we have from (6)

(15) $$g'(\alpha) = \frac{1}{2}f''(\alpha).$$

Taking into account (3), (5) and (15), we obtain from (11)

(16) $$\varphi''(\alpha) = f''(\alpha) - 2\cdot\frac{1}{2}f''(\alpha) = 0.$$

In view of (13), (14) and (16), we conclude that the relations (10) are satisfied for $k = 2$.

For $k \geq 3$, having in mind (3) and (7), we obtain from (12)

(17) $$g^{(r-1)}(\alpha) = 0, \quad \text{for} \quad 2 \leq r \leq k-1$$

and

(18) $$g^{(k-1)}(\alpha) = \frac{1}{k}f^{(k)}(\alpha), \quad \text{for} \quad r = k.$$

On account of (3), (5) and (17), for $r = 1, 2, \cdots, k-1$, from (11) we have, respectively

(19) $$\varphi'(\alpha) = 0, \quad \varphi''(\alpha) = 0, \cdots, \varphi^{(k-1)}(\alpha) = 0.$$

Keeping in mind (3), (5), (17) and (18) for $r = k$, we obtain from (11)

(20) $$\varphi^{(k)}(\alpha) = f^{(k)}(\alpha) - k\cdot\frac{1}{k}f^{(k)}(\alpha) = 0.$$

In view of (13), (14), (19) and (20), we conclude that the relations (10) are satisfied for $k \geq 3$. Since the relations (10) are satisfied for $k = 2$, it follows that they are satisfied for $k \geq 2$, which means that the iterative method (8) is of order $\geq k+1$ for $k \geq 2$.

3. Determination of the Function $g(x)$

From (4), it follows that the function $h(x)$ has the form $h(x) = H(f'(x))$. For $k \geq 2$, in view of (3), the condition $f'(\alpha) = 0$ is satisfied. Therefore, it is not difficult to determine the function of the form $h(x) = H(f'(x))$ for which we have $h(\alpha) = H(f'(\alpha)) = H(0) = 1$.

For every given function $h(x)$, we can obtain from (6) the corresponding function $g(x)$ as

(21) $$g(x) = \frac{1}{k}\int_\alpha^x h(t)f''(t)dt = \frac{1}{k}\int_\alpha^x H\big(f'(t)\big)df'(t) = G\big(f'(x)\big)$$

for which we have $g(\alpha) = G(f'(\alpha)) = G(0) = 0$.

The function $g(x)$ can also be directly given.

4. Some Forms of the Function $g(x)$

Here we give several functions $g(x)$ obtained from (21) for corresponding functions $h(x)$. These are

$$g(x) = \frac{1}{k} f'(x), \quad \text{for} \quad h(x) = 1, \tag{22}$$

which is the result obtained by G. Milovanović [5];

$$g(x) = \frac{1}{k} \frac{f'(x)}{(1 - f'(x))}, \quad \text{for} \quad h(x) = \frac{1}{(1 - f'(x))^2}, \tag{23}$$

which is the result obtained in [7];

$$g(x) = \frac{f'(x)}{k - f'(x)}, \quad \text{for} \quad h(x) = \frac{1}{(1 - \frac{1}{k} f'(x))^2}, \tag{24}$$

which is the result obtained by B. Jovanović [4];

$$g(x) = -\frac{1}{k} \ln\big(1 - f'(x)\big), \quad \text{for} \quad h(x) = \frac{1}{1 - f'(x)}; \tag{25}$$

$$g(x) = \frac{1}{k} \left(e^{f'(x)} - 1\right), \quad \text{for} \quad h(x) = e^{f'(x)}; \tag{26}$$

$$g(x) = e^{\frac{f'(x)}{k}} - 1, \quad \text{for} \quad h(x) = e^{\frac{f'(x)}{k}}; \tag{27}$$

$$g(x) = \frac{1}{k} \sin f'(x), \quad \text{for} \quad h(x) = \cos f'(x); \tag{28}$$

$$g(x) = \frac{1}{k}\big(f'(x) - (f'(x))^2\big), \quad \text{for} \quad h(x) = 1 - 2f'(x). \tag{29}$$

Now we shall give some more forms of the function $g(x)$ for which the conditions (4), (5), (6) and (7) from Theorem 1 are also satisfied. These are

$$g(x) = \frac{1}{k} \left(\left(\frac{stuv - 1 + \left[2 - (1 - sf'(x))^t\right]^u}{stuv} \right)^v - 1 \right), \tag{30}$$

$$g(x) = \frac{1}{k} \left(e^{\frac{[2-(1-sf'(x))^t]^u - 1}{stu}} - 1 \right), \tag{31}$$

$$g(x) = \frac{1}{k} \left(\left(\frac{stv + \ln[2 - (1 - sf'(x))^t]}{stv} \right)^v - 1 \right), \tag{32}$$

$$g(x) = \left(\frac{kstuv - 1 + \left[2 - (1 - sf'(x))^t\right]^u}{kstuv} \right)^v - 1, \tag{33}$$

$$g(x) = e^{\frac{[2-(1-sf'(x))^t]^u - 1}{kstu}} - 1, \tag{34}$$

$$g(x) = \left(\frac{kstv + \ln[2 - (1 - sf'(x))^t]}{kstv} \right)^v - 1, \tag{35}$$

where s, t, u and v are finite parameters $\neq 0$.

We shall consider in particular the function $g(x)$ from (33). In this case Theorem 1 reduces to next theorem.

Theorem 2. *Let* (1) *be an iterative method of order* $k \geq 2$. *Let the function* $f(x)$ *be* $k+1$ *times differentiable in a neighbourhood of the limit point* $x = \alpha$. *Then*

$$
\begin{aligned}
x_{n+1} &= f(x_n) - \left(\left(\frac{kstuv - 1 + \left[2 - (1 - sf'(x_n))^t\right]^u}{kstuv} \right)^v - 1 \right) \cdot (x_n - f(x_n)) = \\
&= x_n - \left(\frac{kstuv - 1 + [2 - (1 - sf'(x_n))^t]^u}{kstuv} \right)^v (x_n - f(x_n)), \\
&\qquad n = 0, 1, 2, \cdots
\end{aligned}
\tag{36}
$$

is an iterative method of arder $\geq k+1$, *where* s, t, u *and* v *are finite parameters* $\neq 0$. *If* $f'(\alpha) \neq 1$, $s = u = 1$ *and* $tv = -1$ *or* $ktv = -1$, *then the method* (36) *holds for* $k \geq 1$.

Proof. On basis of Theorem 1 it folows that Theorem 2 holds for $k \geq 2$. If $f'(\alpha) \neq 1$, $s = u = 1$ and $tv = -1$, the iteration function on the right hand side of (36) reduces to

$$
\varphi(x) = f(x) - \left(\left(\frac{k - 1 + \left(1 - f'(x)\right)^t}{k} \right)^{-\frac{1}{t}} - 1 \right) (x - f(x)). \tag{37}
$$

If $f'(\alpha) \neq 1$, $s = u = 1$ and $ktv = -1$, the iteration function on the right hand side of (36) reduces to

$$
\varphi(x) = f(x) - \left(\left(1 - f'(x)\right)^{-\frac{1}{k}} - 1 \right) \left(x - f(x)\right). \tag{38}
$$

It is not difficult to see that functions (37) and (38) satisfy (10) for $k = 1$, which means that they satisfy it also for $k \geq 1$. This way we have completed the proof of the Theorem 2.

Four parameters, s, t, u and v, stand in the formula (36). Giving these parameters fixed finite values $\neq 0$, one obtains particular iterative formulas.

5. Special Cases of the Formula (36)

5.1. For $s = t = u = v = 1$ and $k \geq 2$, formula (36) reduces to

$$
\begin{aligned}
x_{n+1} &= f(x_n) - \frac{1}{k} f'(x_n) \left(x_n - f(x_n)\right) = \\
&= x_n - \left(1 + \frac{1}{k} f'(x_n)\right) \left(x_n - f(x_n)\right), \quad n = 0, 1, 2, \cdots,
\end{aligned}
\tag{39}
$$

which is the result obtained by G. Milovanović [5].

5.2. For $s = u = 1$, $tv = -1$ and if $f'(\alpha) \neq 1$, the formula (36), which in this case holds for $k \geq 1$, reduces to

$$x_{n+1} = f(x_n) - \left(\left(\frac{k - 1 + \left(1 - f'(x_n)\right)^t}{k} \right)^{-\frac{1}{t}} - 1 \right) \cdot \left(x_n - f(x_n)\right) =$$

(40)

$$= x_n - \left(\frac{k - 1 + \left(1 - f'(x_n)\right)^t}{k} \right)^{-\frac{1}{t}} \left(x_n - f(x_n)\right),$$

$$n = 0, 1, 2, \cdots$$

5.2.1. For $t = 1$, from (40) we obtain

(41)

$$x_{n+1} = f(x_n) - \frac{f'(x_n)}{k - f'(x_n)}\left(x_n - f(x_n)\right) = $$
$$= x_n - \frac{x_n - f(x_n)}{1 - \frac{1}{k} f'(x_n)}, \quad n = 0, 1, 2, \cdots$$

which is the result obtained by B. Jovanović [4].

5.2.2. For $t = -1$, from (40) we have

(42)

$$x_{n+1} = f(x_n) - \frac{1}{k} f'(x_n) \frac{x_n - f(x_n)}{1 - f'(x_n)} =$$
$$= x_n - \left(1 + \frac{1}{k} \frac{f'(x_n)}{1 - f'(x_n)}\right) \left(x_n - f(x_n)\right), \quad n = 0, 1, 2, \cdots$$

which is the result obtained in [7].

5.3. For $s = u = 1$, $ktv = -1$ and if $f'(\alpha) \neq 1$, the formula (36), which in this case holds for $k \geq 1$, reduces to

(43)

$$x_{n+1} = f(x_n) - \left(\left(1 - f'(x_n)\right)^{-\frac{1}{k}} - 1 \right) \left(x_n - f(x_n)\right) =$$
$$= x_n - \frac{x_n - f(x_n)}{\left(1 - f'(x_n)\right)^{\frac{1}{k}}}, \quad n = 0, 1, 2, \cdots$$

6. Examples

If (1) represents Newton's method for finding a simple root $x = \alpha$ of the equation $F(x) = 0$, namely

(44)
$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}, \quad n = 0, 1, 2, \cdots$$

which means that

(45)
$$f(x_n) = x_n - \frac{F(x_n)}{F'(x_n)}$$

and $k = 2$, then we obtain from (36) for $u = 1$ the following method

(46) $$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \left(\frac{2stv + 1 - \left(1 - \frac{sF(x_n)F''(x_n)}{\left(F'(x_n)\right)^2}\right)^t}{2stv} \right)^v,$$

$$n = 0, 1, 2, \cdots$$

According to Theorem 2, the iterative method (46) for fixed finite parameters s, t and v ($stv \neq 0$) is of order ≥ 3, since as we know Newton's method (44) is of order 2.

The asymptotic error constant for the iterative metod (46) is

(47) $$C_3 = \frac{3\left(3 + 2(t-1)s + \frac{1}{v}\right)\left(F''(\alpha)\right)^2 - 4F'(\alpha)F'''(\alpha)}{24\left(F'(\alpha)\right)^2}.$$

6.1. For $s = t = 1$, $v = -1$, we obtain from (46)

(48) $$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \cdot \frac{2\left(F'(x_n)\right)^2}{2\left(F'(x_n)\right)^2 - F(x_n)F''(x_n)}, \quad n = 0, 1, 2, \cdots$$

which is Halley's method (see [2], [3]).

6.2. For $s = 2$, $t = \frac{1}{2}$, $v = -1$, we have from (46) Euler's method (see [3])

(49) $$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \cdot \frac{2}{1 + \left(1 - \frac{2F(x_n)F''(x_n)}{\left(F'(x_n)\right)^2}\right)^{\frac{1}{2}}}, \quad n = 0, 1, 2, \cdots$$

6.3. For $s = t = v = 1$, we obtain from (46)

(50) $$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \cdot \frac{2\left(F'(x_n)\right)^2 + F(x_n)F''(x_n)}{2\left(F'(x_n)\right)^2}, \quad n = 0, 1, 2, \cdots$$

which represents Chebyshev's method (see [1]).

6.4. For $s = \frac{m}{m-1}$, $t = \frac{1}{2}$, $v = -1$, when $F(x)$ is a polynomial of degree $m \geq 2$, we have from (46)

(51) $$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \cdot \frac{m}{1 + (m-1)\left(1 - \frac{m}{m-1} \cdot \frac{F(x_n)F''(x_n)}{\left(F'(x_n)\right)^2}\right)^{\frac{1}{2}}},$$

$$n = 0, 1, 2, \cdots$$

which is the Laguerre method (see [3]).

6.5. For $s = 1$, $t = -\frac{1}{2}$, $v = 1$, we obtain from (46)

$$(52) \qquad x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \left(1 - \frac{F(x_n)F''(x_n)}{\left(F'(x_n)\right)^2}\right)^{-\frac{1}{2}}, \quad n = 0, 1, 2, \cdots$$

which represents Ostrowski's square root method (see [6]).

6.6. For $s = \beta + 1$, $t = \frac{1}{2}$, $v = -1$, we have from (46)

$$(53) \quad x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \cdot \frac{\beta + 1}{\beta + \left(1 - (\beta + 1) \cdot \frac{F(x_n)F''(x_n)}{\left(F'(x_n)\right)^2}\right)^{\frac{1}{2}}}, \quad n = 0, 1, 2, \cdots$$

which represents a one parameter family of iterative formulas obtained by E. Hansen and M. Patrick [3].

6.7. For $s = 1$, $t = -1$, $v = 1$, we have from (46)

$$(54) \qquad x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \cdot \frac{2\left(F'(x_n)\right)^2 - F(x_n)F''(x_n)}{2\left(F'(x_n)\right)^2 - 2F(x_n)F''(x_n)}, \quad n = 0, 1, 2, \cdots$$

which is the method obtained in [7].

6.8. For $s = \frac{m}{m-1}$, $t = \frac{m-1}{m}$, $v = -\frac{1}{2}$, when $F(x)$ is a polynomial of degree $m \geq 2$, we obtain from (46) the method

$$(55) \quad x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \left(1 - \frac{m}{m-1} \cdot \frac{F(x_n)F''(x_n)}{\left(F'(x_n)\right)^2}\right)^{\frac{1-m}{2m}}, \quad n = 0, 1, 2, \cdots$$

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