MATHEMATICA MORAVICA
VOL. 3 (1999), 83–92

GENERALIZATION OF HARDY-LITTLEWOOD-PÓLYA MAJORIZATION PRINCIPLE

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Abstract. This paper continues the study of the general convex functions. In this paper we extension our the former objects of the function in contact and of the function in circled contact.

The following main result is proved: Let $J \subset \mathbf{R}$ be an open interval and let $x_i, y_i \in J (i = 1, \ldots, n)$ be real numbers such that fulfilling

$$x_1 \geq \cdots \geq x_n, \qquad y_1 \geq \cdots \geq y_n. \tag{3}$$

Then, a necessary and sufficient condition in order that

$$\sum_{i=1}^{n} f(x_i) \geq 2 \sum_{i=1}^{n} f(y_i) - n \max \Big\{ f(a), f(b), g\left(f(a), f(b)\right) \Big\} \tag{A}$$

holds for every general convex function $f : J \to \mathbf{R}$ which is in contact with function $g : f(J)^2 \to \mathbf{R}$ and for arbitrary $a, b \in J (a \leq x_i \leq b$ for $i = 1, \ldots, n)$, is that

$$\sum_{i=1}^{k} y_i \leq \sum_{i=1}^{k} x_i \quad (k = 1, \ldots, n-1), \quad \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i. \tag{4}$$

1. Introduction and definitions

This paper continues the study of the general convex fuctions. A function $f : D \to \mathbf{R}$, where $\mathbf{R}$ denotes the real line and D is a convex subset of $\mathbf{R}^n$ is said to be **convex** if

$$f\left(\lambda x + (1 - \lambda) y\right) \leq \lambda f(x) + (1 - \lambda) f(y)$$

for all $x, y \in D$ and for arbitrary $\lambda \in [0, 1]$. Convex functions were introduced (for $n = 1$ and $\lambda = 1/2$) by J. L. Jensen [9], although functions satisfying similar conditions were already treated by O. Hölder [6], J. Hadarmard [3], Ch. Hermite [5] and O. Stolz [19].

G. H. Hardy, J. E. Littlewood and G. Pólya [4] proved in 1929 the following majorization principle for convex funcitions which reads as follows.

AMS (MOS) Subject Classification 1991. Primary: 49A35, 90C30. Secondary: 49A40, 90C48.

Key words and phrases: General convex functions, Convex functions, Inequalities, Majorization principle, Hardy-Littlewood-Pólya majorization principle, Karamata theorem, Characterization of majorization.

Theorem 1. *Let $J \subset \boldsymbol{R}$ be an open interval, let $x_i, y_i \in J (i = 1, \dots, n)$ be real numbers such that fulfilling*

$$x_1 \geq \cdots \geq x_n, \qquad y_1 \geq \cdots \geq y_n, \tag{1}$$

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \quad (k = 1, \dots, n-1), \qquad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i. \tag{2}$$

If $f : J \to \boldsymbol{R}$ is a convex function, then the following inequality holds

$$\sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} f(y_i). \tag{H}$$

Conversely, if for some $x_i, y_i \in J (i = 1, \dots, n)$ such that (1) *holds, and inequality* (H) *is fulfilled for every convex function, then relations* (2) *hold.*

In this paper we consider the preceding facts for general convex functions. In our former paper (see: Tasković [20]) we have introduced the notion of general convex functions. A function $f : D \to \mathbf{R}$, where $\mathbf{R}$ denotes the real line and D is a convex subset of $\mathbf{R}^n$, is said to be **general convex** if there is a function $g : f(D)^2 \to \mathbf{R}$ such that

$$f(\lambda x + (1 - \lambda) y) \leq \max \Big\{ f(x), f(y), g\,(f(x), f(y)) \Big\} \tag{Max}$$

for all $x, y \in D$ and for arbitrary $\lambda \in [0, 1]$. We notice that the set of all convex functions can be proper subset of the set all general convex functions.

In connection with my the former paper [24] in this paper I extension the former objects of the function in contact and of the function in circled contact.

Otherwise, a function $f : D \to \mathbf{R}$ is said to be **general convex with circled contact** if (Max) holds, if there are $x_0, y_0 \in D$ such that $f(x_0) = f(y_0) = g(f(x_0), f(y_0))$ with $g(f(x), f(y)) \leq g(f(x), f(z))$ for $x < y < z$, and if for $x_1 < y_1 < z_1$ the following ineqality

$$\min \Big\{ f(x_1), f(z_1), g(f(x_1), f(z_1)) \Big\} < \tag{Mm}$$

$$< \max \Big\{ \max\,(f(x_1), f(z_1), g(f(x_1), f(z_1)))\,, \max\,(f(y_1), f(z_1), g(f(y_1), f(z_1))) \Big\}$$

implies that there are $a_1, a_2 \in D (a_1 \neq a_2)$ such that

$$f(x_0) < \min \Big\{ f(a_1), f(a_2), g\big(f(a_1), f(a_2)\big) \Big\} \tag{U}$$

or there are $b_1, b_2 \in D (b_1 \neq b_2)$ such that

$$\max \Big\{ f(b_1), f(b_2), g\big(f(b_1), f(b_2)\big) \Big\} < f(x_0). \tag{L}$$

Also, a function $f : D \to \mathbf{R}$ is said to be in **circled contact** with a function $g : f(D)^2 \to \mathbf{R}$ $(g(f(x), f(y)) \leq g(f(x), f(z))$ for $x < y < z)$ if

$f(x_0) = f(y_0) = g(f(x_0), f(y_0))$ for some $x_0, y_0 \in D$, and if (Mm) implies (U) or (L). We notice that the set of all convex functions can be a proper subset of the set all general convex with circled contact functions.

On the other hand, a function $f : D \to \mathbf{R}$ is said to be **general convex with contact** if (Max) holds, if there are $x_0, y_0 \in D$ uch that $f(x_0) = f(y_0) = = g(f(x_0), f(y_0))$, and if

$$\text{(Mc)} \qquad \min\Big\{f(x_1), f(z_1), g(f(x_1), f(z_1))\Big\} < < \max\Big\{f(x_1), f(z_1), g(f(x_1), f(z_1))\Big\}$$

for $x_1 < y_1 < z_1$ implies (U) or (L). Also, a function $f : D \to \mathbf{R}$ is said to be **in contact** with a function $g : f(D)^2 \to \mathbf{R}$, if $f(x_0) = f(y_0) = g(f(x_0), f(y_0))$ for some $x_0, y_0 \in D$, and if (Mc) implies (U) or (L).

We notice that the set of all convex functions can be a proper subset of the set all general convex with contact functions.

In this paper, we prove some inequalities which are characterizations of general convex functions on interval. With the help of the former facts (see: Tasković [20]) in this paper we present a new characterization of general convexity as a majorization principle form an alternative type. With this inequalities alternative we precision and expand our the former majorization principle for general convex functions (see: Theorems 3 and 4 in [24]).

2. Characterizations of general convexity

In the reminder of the paper we consider some statements which give characterizations of general convex functions.

Theorem 2 . (Monotonity of quotients). *Let $J \subset \boldsymbol{R}$ be an open interval, and let $f : J \to \boldsymbol{R}$ be a function in circled contact with a function $g : f(J)^2 \to \to \boldsymbol{R}$. Then each of the following conditions (postulated for every $x, y, z \in J$ with $x < y < z$) is necessary and sufficient for the function f to be general convex with circled contact:*

$$\text{(a)} \qquad f(y) \le \max\{f(x), f(z), g(f(x), f(z))\},$$

$$\text{(b)} \qquad \frac{2f(z) - f(x) - \max\{f(x), f(z), g(f(x), f(z))\}}{z - x} \le \le \frac{2f(z) - f(y) - \max\{f(y), f(z), g(f(y), f(z))\}}{z - y},$$

$$\text{(c)} \qquad \frac{2f(y) - f(x) - \max\{f(x), f(y), g(f(x), f(y))\}}{y - x} \le \le \frac{2f(z) - f(x) - \max\{f(x), f(z), g(f(x), f(z))\}}{z - x}.$$

A variant brief proof of this statement may be found in Tasković [24]. Proof for Theorem 2 is the totally analogous to the proof of Theorem 2 in [24].

In connection with the preceding facts, we are now in a position to formulate our the following characterizations for general convex with contact functions.

Theorem 2a. (Monotonity of quotients). *Let $J \subset \boldsymbol{R}$ be an open interval, and let $f: J \to \boldsymbol{R}$ be a function in contact with a function $g: f(J)^2 \to \boldsymbol{R}$. Then each of the following conditions (postulated for arbitrary $a, b \in J$ and for every $x, y, z \in J$ with $x < y < z$) is necessary and sufficient for the function f to be general convex with contact:*

(a) $$f(\xi) \leq \max\{f(a), f(b), g(f(a), f(b))\} \quad \text{for all} \quad \xi \in [a, b],$$

(b) $$\frac{2f(z) - f(x) - \max\{f(a), f(b), g(f(a), f(b))\}}{z - x} \leq \leq \frac{2f(z) - f(y) - \max\{f(a), f(b), g(f(a), f(b))\}}{z - y},$$

(c) $$\frac{2f(y) - f(x) - \max\{f(a), f(b), g(f(a), f(b))\}}{y - x} \leq \leq \frac{2f(z) - f(x) - \max\{f(a), f(b), g(f(a), f(b))\}}{z - x}.$$

For this statement the proof is analogous to the preceding proof for Theorem 2 based on Theorem 1 (Min-Max Principle) in Tasković [24].

3. Inequalities for general convexity

In this section we give some inequalities which are similar to well known inequality of Hardy-Littlewood-Pólya, i.e., similar with Theorem 1.

We are now in a position to formulate the following fundamental statement for general convex functions.

Theorem 3. (Inequalities alternative). *Let $J \subset \boldsymbol{R}$ be an open interval, let $x_i, y_i \in J$ $(i = 1, \ldots, n)$ be real numbers such that fulfilling*

(3) $$x_1 \geq \cdots \geq x_n, \quad y_1 \geq \cdots \geq y_n,$$

(4) $$\sum_{i=1}^{k} y_i \leq \sum_{i=1}^{k} x_i \quad (k = 1, \ldots, n-1), \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

If $f: J \to \boldsymbol{R}$ is a general convex with circled contact function for some function $g: f(J)^2 \to \boldsymbol{R}$, then either

(A) $$\sum_{i=1}^{n} f(y_i) \leq 2\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} \max\{f(x_i), f(y_i), g(f(x_i), f(y_i))\}$$

or

$$\text{(B)}\qquad \sum_{i=1}^{n} f(x_i) \geq 2\sum_{i=1}^{n} f(y_i) - \sum_{i=1}^{n} \max\{f(x_i), f(y_i), g(f(x_i), f(y_i))\}.$$

Conversely, if for some $x_i, y_i \in J$ $(i = 1, \dots, n)$ such that (3) *holds and inequality* (A) *is fulfilled for every general convex with circled contact function $f : J \to \boldsymbol{R}$, then relations* (4) *hold.*

We notice, proof of this statement is a totally analogous with the proof of Theorem 3 in Tasković [24].

In connection with the preceding statement and Theorem 2a we are now in a position to formulate our main general statement.

Theorem 4. (Majorization Principle). *Let $J \subset \boldsymbol{R}$ be an open interval, let $x_i, y_i \in J$ $(i = 1, \dots, n)$ be real numbers such that fulfilling*

$$\text{(3)}\qquad x_1 \geq \cdots \geq x_n, \quad y_1 \geq \cdots \geq y_n,$$

$$\text{(4)}\qquad \sum_{i=1}^{k} y_i \leq \sum_{i=1}^{k} x_i \qquad (k = 1, \dots, n-1), \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

If $f : J \to \boldsymbol{R}$ is a general convex with contact function for some function $g : f(J)^2 \to \boldsymbol{R}$ then either

$$\text{(N)}\qquad \sum_{i=1}^{n} f(y_i) \leq 2\sum_{i=1}^{n} f(x_i) - n\max\{f(a), f(b), g(f(a), f(b))\}$$

or

$$\text{(M)}\qquad \sum_{i=1}^{n} f(x_i) \geq 2\sum_{i=1}^{n} f(y_i) - n\max\{f(a), f(b), g(f(a), f(b))\}$$

for two arbitrary points $a, b \in J$.

Conversely, if for some $x_i, y_i \in J$ $(i = 1, \dots, n)$ such that (3) *holds and inequality* (N) *or* (M) *is fulfilled for every general convex with contact function $f : J \to \boldsymbol{R}$, then relations* (4) *hold.*

Sketch proof. Adding the same term to both the sides of (N) or (M) does not affect the inequality. Therefore we may a ssume that $x_i \neq y_i$ for $i = 1, \dots, n$. Put

$$D_i := \frac{f(y_i) - 2f(x_i) + \max\{f(a), f(b), g(f(a), f(b))\}}{y_i - x_i}$$

or

$$D_i := \frac{f(x_i) - 2f(y_i) + \max\{f(a), f(b), g(f(a), f(b))\}}{x_i - y_i}$$

for $i = 1, \dots, n$. Since f is a general convex with contact function from Theorem 2a we obtain the inequality (N) or (M).

On the other hand, a brief proof of necessity for the case (N) may be found in Tasković [24]. Thus, we need only show that (M) implies (4). Taking $f(x) = x$ we obtain from (M)

$$\sum_{i=1}^{n} x_i \geq 2 \sum_{i=1}^{n} y_i - n \max\{a, b, g(a,b)\}. \tag{5}$$

If put $a \leq b = y_n$, $g(a,b) = y_n$, then we obtain that the following inequality holds

$$2 \sum_{i=1}^{n} y_i - n \max\{a, b, g(a,b)\} \geq \sum_{i=1}^{n} y_i,$$

i.e., from (5) we obtain an inequality in the last equality in (4). On the other hand, taking $f(x) = -x$ from (M) we obtain, with the preceding relations, together yield the last equality in (4).

Now take an arbitrary $k (1 \leq k < n)$ and $f(x) := \max(0, x - x_k)$. Therefore, putting that is $x = y_i$ $(i = 1, \dots, k)$ and summing up over $i = 1, \dots, k$ we get by (M):

$$\begin{cases} \sum_{i=1}^{k} (y_i - x_k) \leq \sum_{i=1}^{k} f(y_i) \leq \sum_{i=1}^{n} f(y_i) \leq \\ \leq \frac{1}{2} \sum_{i=1}^{n} f(x_i) + \frac{n}{2} \max\{f(a), f(b), g(f(a), f(b))\} \\ = \frac{1}{2} \sum_{i=1}^{k} f(x_i) + \frac{n}{2} \max\{f(a), f(b), g(f(a), f(b))\}. \end{cases} \tag{6}$$

If putting that is $a, b \leq x_k$ and $g(0,0) = 0$, then from (6) we obtain the following inequality

$$\sum_{i=1}^{k} (y_i - x_k) \leq \sum_{i=1}^{k} (x_i - x_k)$$

for $k = 1, \dots, n-1$ whence the first $n-1$ inequalities in (4) follow. Now the proof is finite.

In further let the function $x \mapsto f(x)$ be nonnegative and integrable on (0, 1) so that it is measurable and finite almost everywhere and let $\mu(s)$ be the measure of the set on which $f(x) \geq s$. The function $x \mapsto f^*(x)$ which is inverse to μ is called the decreasing rearrangement of f.

If $x, y \in L^1(0,1)$, we say that y **majorizes** x, in writing $x \prec y$, if

$$\int_0^s x^*(t)dt \leq \int_0^s y^*(t)dt \quad \text{for} \quad 0 < s < 1,$$

and

$$\int_0^1 x(t)dt = \int_0^1 y(t)dt.$$

We shall now give an integral inequality, which is connected with the majorization of functions, and which is analogue the preceding result, because without proof. We note G.H. Hardy, J.E. Littlewood and G. Pólya also proved in [4] an integral analogy of the inequality which appears in the their majorization principle.

Theorem 4a. (Integral analogue of majorization). *The following inequality of the form*

$$\int_0^1 f(y(t))dt \leq 2\int_0^1 f(x(t))dt - \max\{f(a), f(b), g(f(a), f(b))\}$$

or

$$\int_0^1 f(x(t))dt \geq 2\int_0^1 f(y(t))dt - \max\{f(a), f(b), g(f(a), f(b))\}$$

holds for some function $g : [0,1]^2 \to \boldsymbol{R}$, *for arbitrary points* $a, b \in [0,1]$, *and for any general convex with contact function* f *if and only if* x *majorizes* y.

With this preceding statements we precision and expand our the former majorization principles for general convex with contact functions (Theorems 4 and 4a in [24]).

4. Some consequences

On the other hand, if to teasing on the convex class of functions taking

$$\text{(C)} \qquad \max\Big\{f(x_i), f(y_i), g(f(x_i), f(y_i))\Big\} = \lambda f(x_i) + (1-\lambda) f(y_i)$$

for arbitrary $\lambda \in [0,1]$ in Theorem 3, then from inequalities (A) and (B) we obtain the preceding Theorem 1 of Hardy-Littlewood-Pólya.

This means that Theorem 3 extends Theorem 1 to general convex functions. Also, from Theorem 3 as an immediate consequence we obtain and a correspond statement for quasiconvex functions.

In connection with the preceding facts, since inequality (N) or (M) for $a \leq x_i \leq b$ $(i = 1, \ldots, n)$ is equivalent only to inequality (M), thus we can Theorem 4 write in the following equivalent form in this case.

Theorem 4b. *Let* $J \subset \boldsymbol{R}$ *be an open interval and let* $x_i, y_i \in J$ $(i = 1, \ldots, n)$ *be real numbers such that fulfilling*

$$\text{(3)} \qquad x_1 \geq \cdots \geq x_n, \qquad y_1 \geq \cdots \geq y_n.$$

Then, a necessary and sufficient condition in order that

$$\text{(M)} \qquad \sum_{i=1}^{n} f(x_i) \geq 2\sum_{i=1}^{n} f(y_i) - n \max\{f(a), f(b), g(f(a), f(b))\}$$

holds for every general convex function $f : J \to \boldsymbol{R}$ which is in contact with function $g : f(J)^2 \to \boldsymbol{R}$ and for arbitrary $a, b \in J$ ($a \leq x_i \leq b$ for $i = = 1, \ldots, n$), is that

$$\text{(4)} \qquad \sum_{i=1}^{k} y_i \leq \sum_{i=1}^{k} x_i \quad (k = 1, \ldots, n-1), \quad \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i.$$

As an imediate consequence of Theorem 3, directly, we obtain the following inequality. Indeed, putting in (B) $y_1 = \ldots = y_n = n^{-1}\sum_{i=1}^{n} x_i$ we get

$$\text{(Gc)} \qquad \begin{aligned} \sum_{i=1}^{n} f(x_i) \geq 2nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) - \\ - \sum_{i=1}^{n} \max\left\{ f(x_i), f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right), g\left(f(x_i), f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right)\right\} \end{aligned}$$

for every general convex function $f : J \to \mathbf{R}$ ($J \subset \mathbf{R}$ is an open interval) which is in circled contact with function $g : f(J)^2 \to \mathbf{R}$.

This inequality is a generalization of Jensen's inequality for convex functions. Indeed, if to teasing on the convex class of functions taking (C), then from (Gc) we get Jensen's inequality.

The following statement is very similar to Theorem 3.

Theorem 3a. *Let $J \subset \boldsymbol{R}$ be an open interval, let $x_i, y_i \in J$ ($i = = 1, \ldots, n$) be real numbers such that fulfilling* (3) *and such that*

$$\text{(I)} \qquad \sum_{i=1}^{k} y_i \leq \sum_{i=1}^{k} x_i \qquad (k = 1, \ldots, n).$$

If $f : J \to \boldsymbol{R}$ is an increasing general convex with circled contact function for some function $g : f(J)^2 \to \boldsymbol{R}$ then either

$$\text{(A)} \qquad \sum_{i=1}^{n} f(y_i) \leq 2\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} \max\{f(x_i), f(y_i), g(f(x_i), f(y_i))\}$$

or

$$\text{(B)} \qquad \sum_{i=1}^{n} f(x_i) \geq 2\sum_{i=1}^{n} f(y_i) - \sum_{i=1}^{n} \max f(x_i), f(y_i), g(f(x_i), f(y_i))\}.$$

Conversely, if for some $x_i, y_i \in J$ $(i = 1, \dots, n)$ *such that* (3) *holds and inequality* (A) *is fulfilled for every increasing general convex with circled contact function* $f : J \to \boldsymbol{R}$, *then inequalities* (I) *hold.*

This proof of this statement is totally analogous to the preceding proof of Theorem 3. The following statement is very similar to Theorem 4b.

Theorem 4c. *Let* $J \subset \boldsymbol{R}$ *be an open interval and let* $x_i, y_i \in J$ $(i = 1, \dots, n)$ *be real numbers such that fulfilling*

$$x_1 \geq \cdots \geq x_n, \qquad y_1 \geq \cdots \geq y_n. \tag{3}$$

Then, a necessary and sufficient condition in order that

$$\sum_{i=1}^{n} f(x_i) \geq 2 \sum_{i=1}^{n} f(y_i) - n \max \{f(a), f(b), g(f(a), f(b))\} \tag{M}$$

holds for every increasing general convex function $f : J \to \boldsymbol{R}$ *which is in contact with function* $g : f(J)^2 \to \boldsymbol{R}$ *and for arbitrary* $a, b \in J$ $(a \leq x_i \leq b$ *for* $i = 1, \dots, n)$, *is that*

$$\sum_{i=1}^{k} y_i \leq \sum_{i=1}^{k} x_i \qquad (k = 1, \dots, n). \tag{I}$$

The proof of this statement is very similar and a totally analogous to the preceding proof of Theorem 4.

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Received July 15, 1998.