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CHARACTERIZATION OF GENERAL CONVEX FUNCTIONS AND ITS APPLICATIONS

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Abstract. In this paper we continue the study of the general convex functions, which are introduced in our former paper (Tasković, Math. Japonica, **37** (1992), 367-372). This paper present a new characterization of general convex functions in term of general level sets. Applications in convex analysis are considered.

1. Introduction and main result

In our former paper, Tasković [5], have introduced the notion of general convex functions. A function $f : D \to \mathbb{R}$, where $\mathbb{R}$ denotes the real line and D is a convex subset of $\mathbb{R}^n$, is said to be **general convex** if there is a function $g : f(D)^2 \to \mathbb{R}$ such that

$$\text{(Max)} \qquad f(\lambda x + (1-\lambda)y) \le \max\Big\{f(x),\ f(y),\ g\big(f(x),\ f(y)\big)\Big\}$$

for all $x, y \in D$ and for arbitrary $\lambda \in [0,1]$. We notice that the set of all convex and quasiconvex function can be a proper subset of the set all general convex functions.

In order, the function $g : \mathbb{R}^2 \to \mathbb{R}$ is **increasing** if $x_i, y_i \in \mathbb{R}$ and $x_i \le y_i$ $(i = 1, 2)$ implies $g(x_1, x_2) \le g(y_1, y_2)$. On the other hand, the function $g : \mathbb{R}^2 \to \mathbb{R}$ is **level increasing** if it is increasing and with the property

$$g\left(\max\{x, g(x,x)\}, \max\{x, g(x,x)\}\right) \le \max\{x, g(x,x)\}$$

for every $x \in \mathbb{R}$.

It is well-known that a convex function can be characterized by convexity of its epigraph. Also, we know that a quasiconvex function can be characterized by convexity of its level sets.

In this paper we present a new characterization of general convex functions as convexity of their general level sets. In this sense, we are now in a position to formulate main general statement.

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Theorem 1. *Let $D \subset \mathbb{R}^n$ be a convex and open set. The function $f : D \to \mathbb{R}$ is general convex for some level increasing function $g : \mathbb{R}^2 \to \mathbb{R}$ if and only if*

$$\text{(G1)} \quad g(D_a) := \Big\{x \in D \,|\, \max\{f(x), g(f(x), f(x))\} \le \max\{a, g(a,a)\}\Big\}$$

is a convex set for each number $a \in \mathbb{R}$.

Proof. Suppose that f is a general convex function, and let $x, y \in g(D_a)$. Therefore $x, y \in D$ and

$$\text{(1)} \quad \max\{f(x), g(f(x), f(x))\}, \ \max\{f(y), g(f(y), f(y))\} \le \max\{a, g(a,a)\}.$$

Let $z = \lambda x + (1-\lambda)y$ for $\lambda \in [0,1]$. By convexity of D we obtain $z \in D$. Furhemore, by general convexity of f, i.e., from (Max) and (1) we have

$$f(z) \le \max\Big\{f(x), f(y), g(f(x), f(y))\Big)\Big\} \le$$

$$\le \max\Big\{f(x), f(y), \max\big(g(f(x), f(x)), g(f(y), f(y))\big)\Big\} \le \max\{a, g(a,a)\}.$$

Thus $f(z) \le \max\{a, g(a,a)\}$ and from level increasing of $g : \mathbb{R}^2 \to \mathbb{R}$ we obtain $g(f(z), f(a)) \le g(\max\{a, g(a,a)\}, \max\{a, g(a,a)\}) \le \max\{a, g(a,a)\}$. This mean that is $\max\{f(z), g(f(z), f(z))\} \le \max\{a, g(a,a)\}$, i.e., $z \in g(D_a)$. Thus $g(D_a)$ is a convex set.

Conversely, suppose that $g(D_a)$ is a convex set for each number $a \in \mathbb{R}$. Let $z = \lambda x + (1-\lambda)y$ for all $\lambda \in [0,1]$. Notice that $x, y \in g(D_a)$ for

$$\max\{a, g(a,a)\} = \max\{f(x), f(y), g(f(x), f(y))\}.$$

By assumption, $g(D_a)$ is convex, so that $z \in g(D_a)$. Therefore,

$$f(z) \le \max\Big\{f(z), g(f(z), f(z))\Big\} \le \max\Big\{a, g(a,a)\Big\} =$$

$$= \max\Big\{f(x), f(y), g(f(x), f(y))\Big\}.$$

Hence, f is a general convex function. The proof is complete.

We notice, from the preceding proof of Theorem 1 as an immediate fact we obtain the following statement.

Corollary 1. *Let $D \subset \mathbb{R}^n$ be a convex and open set, and let $f : D \to \mathbb{R}$. If there is a function $g : \mathbb{R}^2 \to \mathbb{R}$ such that the sets $g(D_a)$ are convex, then f is a general convex function.*

On the other hand, from the preceding statement, we are now in a position to formulate the following consequence for quasiconvex functions.

In this sense, a function $f : D \to \mathbb{R}$, where D is a convex subset of $\mathbb{R}^n$, is said to be **quasiconvex** if

$$f(\lambda x + (1-\lambda)y) \le \max\{f(x), f(y)\}$$

for all $x, y \in D$ and for arbitrary $\lambda \in [0,1]$. We notice that the set of all quasiconvex functions can be a proper subset of the set all general convex functions.

Corollary 2. (De Finetti [1], Fenchel [2]). *Let $D \subset \mathbb{R}^n$ be a convex and open set. The function $f : D \to \mathbb{R}$ is quasiconvex if and only if*

$$L_a := \{x \in D \big| \ f(x) \le a\}$$

*is a convex set for each number $a \in \mathbb{R}$. (The set L_a is called **level set**.)*

Proof. If to teasing on the quasiconvex class functions taking that $g(f(x),\ f(y)) = \max\{f(x), f(y)\}$ from Theorem 1 we obtain directly this statement for quasiconvex functions and level sets. The proof is complete.

Further, as an immediate consequence of Theorem 1 we obtain directly the following statement with which we precision Lemma 1 of [5].

Corollary 3. (Extremal Principle). *Let X be a reflexive Banach space and let M be a nonempty, closed, bounded and convex set in X. If $f : M \to$ $\to \mathbb{R} \cup \{+\infty\}$ is a general convex function for some continuous level increasing function $g : \mathbb{R}^2 \to \mathbb{R}$ and if the set $g(D_a)$ is closed for all $a \in \mathbb{R}$, then f has a minimum on M.*

Proof. The set M is weakly compact, because M is bounded, closed and convex set in reflexive Banach space X. Further, $g(D_a)$ is closed and convex (from Theorem 1), and hence weakly closed. Therefore f is lower semicontinuous in the weak topology on M. The conclusion now follows from Weierstrass theorem. The proof is complete.

2. Further applications

We now give a result which shows that the maximum of a general convex function over a compact polyhedral set occurs at an extreme point.

A nonempty set $D \subset \mathbb{R}^n$ is called a **polyhedral set** if it is the intersection of a finite number of closed half spaces. Note that a polyhedral set is a closed convex set. A vector $z \in D$ is called an **extreme point** of D if $z = \lambda x + (1 - \lambda)y$ with $\lambda \in (0,1)$ and $x, y \in D$ implies that $z = x = y$.

Theorem 2. *Let $D \subset \mathbb{R}^n$ be a nonempty compact polyhedral set, and let $f : D \to \mathbb{R}$ be a continuous and general convex function for some level increasing function $g : \mathbb{R}^2 \to \mathbb{R}$. Consider the problem to maximize $x \mapsto f(x)$ subject to $x \in D$. Then there exists an optimal solution $\xi \in D$ to the problem which is an extreme point of D.*

Proof. Note that f is continuous on D and hence attains a maximum, say, at $\xi \in D$. If there is an extreme point whose objective is equal to $f(\xi)$, then the result is at hand. Otherwise, let $x_1, \dots, x_k$ be the extreme points of D, and assume that $f(\xi) > f(x_j)$ for $j = 1, \dots, k$. By representation of points in D, $\xi \in D$ can be represented as $\xi = \lambda_1 x_1 + \cdots + \lambda_k x_k$, where $\lambda_1 + \cdots + \lambda_k = 1$ for $\lambda_j \geq 0$ $(j = 1, \dots, k)$. Since $f(\xi) > f(x_j)$ for each $j = 1, \dots, k$ we obtain

$$f(\xi) > \max_{j=1,\dots,k} f(x_j) := \max\{a, g(a,a)\}. \tag{2}$$

Now consider the sets $g(D_a)$ with (G1) for some level increasing function $g : \mathbb{R}^2 \to \mathbb{R}$. Note that $x_j \in g(D_a)$ for $j = 1, \dots, k$ and by general convexity of f (Theorem 1) the set $g(D_a)$ is convex. Hence, $\xi = \lambda_1 x_1 + \cdots + \lambda_k x_k$ belongs to $g(D_a)$, i.e.,

$$\max\{f(\xi), g\big(f(\xi), f(\xi)\big)\} \leq \max\{a, g(a,a)\}.$$

This implies that $f(\xi) \leq \max\{a, g(a,a)\}$ which contradicts (2). This contradiction shows that $f(\xi) = f(x_j)$ for some extreme point x_j. The proof is complete.

We notice that quasiconvex functions are, de facto, general convex functions. Thus we obtain directly as an immediate consequence of Theorem 2 and corresponding result for quasiconvex functions. This mean that the maximum of a quasiconvex function over a compact polyhedral set occurs at an extreme point.

3. General level sets

In what follows we assume that D is a nonempty convex subset of $\mathbb{R}^n$ and ε is a positive constant. Recall that a function $f : D \to \mathbb{R}$ is said to be **ε-quasiconvex** if

$$f\big(\lambda x + (1-\lambda)y\big) \leq \max\{f(x), f(y)\} + \varepsilon$$

for all $x, y \in D$, and all $\lambda \in [0,1]$. For $\varepsilon = 0$ this definition reduces to that of **quasiconvex function**, cf. Roberts-Varberg [4].

Recall that a function $f : D \to \mathbb{R}$ is said to be **ε-general convex** if for some $\varepsilon > 0$ there is a function $g : f(D)^2 \to \mathbb{R}$ such that

$$f\big(\lambda x + (1-\lambda)y\big) \leq \max\Big\{f(x), f(y), g\big(f(x), f(y)\big)\Big\} + \varepsilon \tag{M}$$

for all $x, y \in D$ and for all $\lambda \in [0,1]$. For $\varepsilon = 0$ this definition reduces to that of **general convex function**.

On the other hand, the function $g : \mathbb{R}^2 \to \mathbb{R}$ is **ε-level increasing** if it is increasing and with the property

$$g\left(\max\{x, g(x,x)\} + \varepsilon, \max\{x, g(x,x)\} + \varepsilon\right) \leq \max\{x, g(x,x)\} + \varepsilon$$

for every $x \in \mathbb{R}$ and $\varepsilon > 0$.

Assume that $f : D \to \mathbb{R}$ is a ε-general convex function for some ε-level increasing function $g : \mathbb{R}^2 \to \mathbb{R}$ and consider the general level sets

$$g(L_a) := \left\{x \in D \middle| \max\left\{f(x), g\big(f(x), f(x)\big)\right\} \leq a\right\}$$

for $a \in \mathbb{R}$. It is clear that $\cup_{a\in\mathbb{R}} g(L_a) = D$ and $g(L_a) \subset g(L_b)$ whenever $a \leq b$. We notice, the set $g(L_a)$ is called **general level set**.

We are now in a position to formulate the following statement with which we precision and expand a fact (a comment) in [5].

Theorem 3. *Let $D \subset \mathbb{R}^n$ be a nonempty convex set, and let $f : D \to \mathbb{R}$ be a ε-general convex function for some ε-level increasing function $g : \mathbb{R}^2 \to \to \mathbb{R}$. If $x_1, \dots, x_{m+1} \in g(L_a)$ for $m \in \mathbb{N}$, $a \in \mathbb{R}$ and $\lambda_1 + \cdots + \lambda_{m+1} = 1$, ($\lambda_1, \dots, \lambda_{m+1} \in [0,1]$), then*

$$\lambda_1 x_1 + \cdots + \lambda_{m+1} x_{m+1} \in g\left(L_{\max\{a, g(a,a)\} + \varepsilon k(m)}\right),$$

where $k(m) = 1 + [\log_2 m]$.

Proof. If $x, y \in g(L_a)$ and $\lambda_1 + \lambda_2 = 1$ ($\lambda_1, \lambda_2 \in [0,1]$) we have $\max\{f(x),\ g\big(f(x), f(x)\big)\} \leq a$, and $\max\{f(y), g\big(f(y), f(y)\big)\} \leq a$. From inequality (M) for $z = \lambda_1 x + \lambda_2 y$ we obtain

$$f(z) \leq \max\left\{f(x), f(y), g\big(f(x), f(y)\big)\right\} + \varepsilon \leq \max\{a, g(a,a)\} + \varepsilon.$$

By ε-level increasing of $g : \mathbb{R}^2 \to \mathbb{R}$ we obtain

$$g\big(f(z), f(z)\big) \leq$$
$$\leq g\Big(\max\{a, g(a,a)\} + \varepsilon, \max\{a, g(a,a)\} + \varepsilon\Big) \leq \max\{a, g(a,a)\} + \varepsilon.$$

This means that $\max\{f(z), g(f(z), f(z))\} \leq \max\{a, g(a,a)\} + \varepsilon$, i.e., $z = \lambda_1 x + \lambda_2 y \in g\left(L_{\max\{a,g(a,a)\}+\varepsilon}\right)$. By induction we can show that

$$\lambda_1 x_1 + \cdots + \lambda_{2^r} x_{2^r} \in g\left(L_{\max\{a,g(a,a)\}+\varepsilon r}\right) \tag{3}$$

for all $r \in \mathbb{N}$, for $x_1, \dots, x_{2^r} \in D$ and $\lambda_1, \dots, \lambda_{2^r} \in [0,1]$ with $\lambda_1 + \cdots + \lambda_{2^r} = 1$. Fix an $m \in \mathbb{N}$ and assume that $x_1, \dots, x_m \in D$ with $\lambda_1, \dots, \lambda_m \in [0,1]$ and $\lambda_1 + \cdots + \lambda_m = 1$. Take the minimal $r \in \mathbb{N}$ such that $m + 1 \leq 2^r$. One can easily check that $r = [\log_2 m] + 1 := k(m)$. In the case $m + 1 < 2^r$, let us put $\lambda_{m+2} = \cdots = \lambda_{2^r} = 0$ and $x_{m+2} = \cdots = x_{2^r} := x_1$. Then by preceding facts and (3), we obtain

$$\lambda_1 x_1 + \cdots + \lambda_{m+1} x_{m+1} =$$
$$= \lambda_1 x_1 + \cdots + \lambda_{2^r} x_{2^r} \in g\left(L_{\max\{a,g(a,a)\}+\varepsilon k(m)}\right).$$

The proof is complete.

From Theorem 3 we are now in a position to formulate the following directly consequence for quasiconvex functions.

Corollary 4. (Nikodem [3]). *Let $D \subset \mathbb{R}^n$ be a nonempty convex set, and let $f : D \to \mathbb{R}$ be a ε-quasiconvex function. If $x_1, \dots, x_{m+1} \in L_a$ for $m \in \mathbb{N}$, $a \in \mathbb{R}$ and $\lambda_1 + \cdots + \lambda_{m+1} = 1$ $(\lambda_1, \dots, \lambda_{m+1} \in [0,1])$, then*

$$\lambda_1 x_1 + \cdots + \lambda_{m+1} x_{m+1} \in L_{a+\varepsilon k(m)},$$

for $k(m) := 1 + [\log_2 m]$.

Proof. If to teasing on the ε-quasiconvex class functions taking that $g(f(x), f(y)) = \max\{f(x), f(y)\}$ from Theorem 3 we obtain directly this statement, because in this case $g(L_a) = L_a$. The proof is complete.

4. References

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