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NOTE ON (n, m)-GROUPS

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Abstract. Among the results of the paper is the following proposition. Let $2m \leq n < 3m$ and let (Q, A) be an (n, m)-groupoid $(n, m \in N)$. Then, (Q, A) is an (n, m)-group iff there are mappings $^{-1}$ and $\mathbf{e}$ respectively of the sets Q^{n-m} and Q^{n-2m} into the set Q^m such that the following laws hold in the algebra $(Q, A, ^{-1}, \mathbf{e})$:

$$A\left(A\left(x_1^n\right), x_{n+1}^{2n-m}\right) = A\left(x_1, A\left(x_2^{n+1}\right), x_{n+2}^{2n-m}\right),$$
$$A\left(A\left(x_1^n\right), x_{n+1}^{2n-m}\right) = A\left(x_1^{n-m}, A\left(x_{n-m+1}^{2n-m}\right)\right),$$
$$A\left(x_1^m, a_1^{n-2m}, \mathbf{e}\left(a_1^{n-2m}\right)\right) = x_1^m \quad \text{and}$$
$$A\left(x_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) = \mathbf{e}\left(a_1^{n-2m}\right).$$

1. Introduction

1.1. Definitions. *Let $n \geq m+1$ $(n, m \in N)$ and (Q, A) be an (n, m)-groupoid $(A : Q^n \to Q^m)$. Then: (a) we say that (Q, A) is an (n, m)-***semigroup** *iff for every $i, j \in \{1, \ldots, n-m+1\}, i < J$, $< i, j$, the following law holds*

$$A\left(x_1^{i-1}, A\left(x_i^{i+n-1}\right), x_{i+n}^{2n-m}\right) = A\left(x_1^{j-1}, A\left(x_j^{j+n-1}\right), x_{j+n}^{2n-m}\right)$$

*[:< 1, j > associative law]; and (b) we say that (Q, A) is an (n, m)-***group** *iff (Q, A) is an (n, m)-semigroup and for every $a_1^n \in Q$ there is* **exactly one** *sequence x_1^m over Q and* **exactly one** *sequence y_1^m over Q such that the following equalities hold*

$$A\left(a_1^{n-m}, x_1^m\right) = a_{n-m+1}^n \quad \text{and} \quad A\left(y_1^m, a_1^{n-m}\right) = a_{n-m+1}^n.$$

(See, also [3].)

1.2. Remark. *A notion of an (n, m)-group was introduced by G. Čupona in [2] as a generalization of the notion of a group (n-group – [1]). The peper [3] is mainly a survey on the known results for vector valued groupoids, semigroups and groups (to 1988).* □

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1.3. Definition [5]: *Let $n \geq 2m$ and let (Q, A) be an (n,m)-groupoid. Let also* $\mathbf{e}$ *be mappings of the set Q^{n-2m} into the set Q^m. Then, we say that* $\mathbf{e}$ *is an* $\{1, n-m+1\}$**-neutral operation** *of the (n,m)-groupoid (Q, A) iff for every $a_1^{n-2m}, x_1^m \in Q$ the following equalities hold:*

$$A\left(x_1^m, a_1^{n-2m}, \mathbf{e}\left(a_1^{n-2m}\right)\right) = x_1^m \quad \text{and} \quad A\left(\mathbf{e}\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right) = x_1^m.$$

1.4. Remark: *Every (n,m)-groupoid ($n \geq 2m$) has* **at most one** $\{1, n-m+1\}$**-neutral operation** *($\{i,j\}$-neutral operation) [:[5]]. For $(n,m) = (2,1)$, $\mathbf{e}(a_1^\circ)(= \mathbf{e}(\emptyset))$ is a neutral element of the groupoid (Q, A). See, also [4,7].* □

2. Auxiliary propositions

In this paper the following $< I, J$-associative laws have the prominence:

(1) $$A\left(A\left(x_1^n\right), x_{n+1}^{2n-m}\right) = A\left(x_1^{n-m}, A\left(x_{n-m+1}^{2n-m}\right)\right),$$

(1L) $$A\left(A\left(x_1^n\right), x_{n+1}^{2n-m}\right) = A\left(x_1, A\left(x_2^{n+1}\right), x_{n+2}^{2n-m}\right),$$

(1Lm) $$A\left(A\left(a_1^m, b_1^{n-m}\right), c_1^m, d_1^{n-2m}\right) = A\left(a_1^m, A\left(b_1^{n-m}, c_1^m\right), d_1^{n-2m}\right),$$

(1R) $$A\left(x_1^{n-m-1} A\left(x_{n-m}^{2n-m-1}\right), x_{2n-m}\right) = A\left(x_1^{n-m}, A\left(x_{n-m+1}^{2n-m}\right)\right) \quad \text{and}$$

(1Rm) $$A\left(a_1^{n-2m}, A\left(b_1^m, c_1^{n-m}\right), d_1^m\right) = A\left(a_1^{n-2m}, b_1^m, A\left(c_1^{n-m}, d_1^m\right)\right). \square$$

2.1. Proposition: *Let $n \geq 2m$ and let (Q, A) be an (n,m)-groupoid. Further on, let the $< 1, n-m+1 >$-associative law [:(1)] holds in (Q, A) and let for every $a_1^n \in Q$ there is* **at least one** *sequence x_1^m over Q and* **at least one** *sequence y_1^m over Q such that the following equalities hold*

$$A\left(a_1^{n-m}, x_1^m\right) = a_{n-m+1}^n \quad \text{and} \quad A\left(y_1^m, a_1^{n-m}\right) = a_{n-m+1}^n.$$

Then, there are mappings $\mathbf{e}$ *and* $^{-1}$ *repectively of the sets Q^{n-2m} and Q^{n-m} into the set Q^m such that the following laws hold in the algebra of the form $(Q, \{A, ^{-1}, \mathbf{e}\})$*

(2L) $$A\left(\mathbf{e}\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right) = x_1^m,$$

(2R) $$A\left(x_1^m, a_1^{n-2m}, \mathbf{e}\left(a_1^{n-2m}\right)\right) = x_1^{m\,1},$$

(3L) $$A\left(\left(a_1^{n-2m}, x_1^m\right)^{-1}, a_1^{n-2m}, x_1^m\right) = \mathbf{e}\left(a_1^{n-2m}\right),$$

(3R) $$A\left(x_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) = \mathbf{e}\left(a_1^{n-2m}\right),$$

(4L) $$A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, A\left(b_1^m, a_1^{n-2m}, x_1^m\right)\right) = x_1^m, \quad \text{and}$$

(4R) $$A\left(A\left(x_1^m, a_1^{n-2m}, b_1^m\right), a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right) = x_1^m.$$

Proof. The following statements hold

1° (Q,A) has a $\{1,n-m+1\}$ - neutral operation.

2° The $< 1,2n-2m+1 >$ - associative law holds in $(Q,\overset{2}{A})$, where

$$\overset{2}{A}\left(x_1^{2n-m}\right) \overset{det}{=} A\left(A\left(x_1^n\right), x_{n+1}^{2n-m}\right),$$

and for every $a_1^{2n-m} \in Q$ there is **at least one** sequence x_1^m over Q and **at least one** sequence y_1^m over Q such that the following equalities hold

$$\overset{2}{A}(a_1^{2n-2m}, x_1^m) = a_{2n-2m+1}^{n-m} \quad \text{and} \quad \overset{2}{A}(y_1^m, a_1^{2n-2m}) = a_{2n-2m+1}^{2n-m}.$$

3° $(Q,\overset{2}{A})$ has a $\{1,2n-2m+1\}$-neutral operation.

The proof of 1° :

Let b_1^m be an arbitrary (fixed) sequence over Q. Then for every sequence a_1^{n-2m} over Q there is **at least one** $\mathbf{e}_L(a_1^{n-2m}) \in Q^m$ such that the following equality holds

(a) $$A\left(\mathbf{e}_L\left(a_1^{n-2m}\right), a_1^{n-2m}, b_1^m\right) = b_1^m.$$

On the other hand, for every sequence c_1^m over Q and for every sequence k_1^{n-2m} over Q there is **at least one** sequence t_1^m over Q such that the following equality holds

(b) $$c_1^m = A\left(b_1^m, k_1^{n-2m}, t_1^m\right).$$

By $(a),(b)$ and the assumption that the $< 1, n-m+1 >$-associative law holds in (Q,A), we conclude that the following series equalities hold:

$$A\left(\mathbf{e}_L\left(a_1^{n-2m}\right), a_1^{n-2m}, c_1^m\right) = A\left(\mathbf{e}_L\left(a_1^{n-2m}\right), a_1^{n-2m}, A\left(b_1^m, k_1^{n-2m}, t_1^m\right)\right) =$$
$$A\left(A\left(\mathbf{e}_L\left(a_1^{n-2m}\right), a_1^{n-2m}, b_1^m\right), k_1^{n-2m}, t_1^m\right) =$$
$$A\left(b_1^m, k_1^{n-2m}, t_1^m\right) = c_1^m,$$

whence we conclude that for every sequence c_1^m over Q and for every sequence a_1^{n-2m} over Q the following equality holds

$$A\left(\mathbf{e}_L\left(a_1^{n-2m}\right), a_1^{n-2m}, c_1^m\right) = c_1^m,$$

i.e., that (Q,A) has (at least one) **left** $\{1,n-m+1\}$-neutral operation $\mathbf{e}_L$ /:[5]/. Similary, it is possible to prove that there is a **right** $\{1,n-m+1\}$-neutral operation $\mathbf{e}_R$ in (Q,A) /:[5]/. Thus, for every sequence a_1^{n-2m} over Q the following equalities hold

$$A\left(\mathbf{e}_L\left(a_1^{n-2m}\right), a_1^{n-2m}, \mathbf{e}_R\left(a_1^{n-2m}\right)\right) = \mathbf{e}_R\left(a_1^{n-2m}\right) \quad \text{and}$$
$$A\left(\mathbf{e}_L\left(a_1^{n-2m}\right), a_1^{n-2m}, \mathbf{e}_R\left(a_1^{n-2m}\right)\right) = \mathbf{e}_L\left(a_1^{n-2}\right),$$

whence $\mathbf{e}_L = \mathbf{e}_R \; (= \mathbf{e})$.

The sketch of the proof of 2° :

1) $\overset{2}{A}(\overset{2}{A}(x_1^n, u_1^{n-2m}, v_1^m), y_{m+1}^{n-m}, y_{n-m+1}^{n}, y_{n+1}^{2n-m}) =$
$A(A(A(A(x_1^n), u_1^{n-2m}, v_1^m), y_{m+1}^{n-m}, y_{n-m+1}^{n}), y_{n+1}^{2n-m}) =$
$A(A(A(x_1^n), u_1^{n-2m}, v_1^m), y_{m+1}^{n-m}, A(y_{n-m+1}^{2n-m})) =$
$A(A(x_1^n), u_1^{n-2m}, A(v_1^m, y_{m+1}^{n-m}, A(y_{n-m+1}^{2n-m}))) =$
$A(A(x_1^n), u_1^{n-2m}, A(A(v_1^m, y_{m+1}^{n}), y_{n+1}^{2n-m})) =$
$\overset{2}{A}(x_1^n, u_1^{n-2m}, \overset{2}{A}(v_1^m, y_{m+1}^{n}, y_{n+1}^{2n-m}))$;

2) $\overset{2}{A}(a_1^{2n-2m}, x_1^m) = a_{2n-2m+1}^{2n-m} \Leftrightarrow$
$A(A(a_1^n), a_{n+1}^{2n-2m}, x_1^m) = a_{2n-2m+1}^{2n-m}$ and
$\overset{2}{A}(y_1^m, a_1^{2n-2m}) = a_{2n-2m+1}^{2n-m} \Leftrightarrow$
$A(y_1^m, a_1^{n-2m}, A(a_{n-2m+1}^{2n-2m})) = a_{2n-2m+1}^{2n-m}$.

The proof of 3° :

By 1° and 2°, we conclude that the $(2n-m, m)$-groupoid $(Q, \overset{2}{A})$ has an $\{1, 2n-2m+1\}$-neutral operation (let it be denoted by) E.

Let

$$\left(a_1^{n-2m}, x_1^m\right)^{-1} \overset{def}{=} \mathsf{E}\left(a_1^{n-2m}, x_1^m, a_1^{n-2m}\right).$$

Hence, by 1° – 3°, we conclude that the laws (2L)–(4L) and (2R)–(4R) hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$. (See, also [6,7],).

2.2. Proposition: *Let $n > m+1$ and let (Q, A) be an (m,n)-groupoid. Also let*

(a) the (1L) [(1R)] law holds in (Q, A); and

(b) for every $x_1^m, y_1^m, a_1^{n-m} \in Q$ the following implication holds

$$A\left(x_1^m, a_1^{n-m}\right) = A\left(y_1^m, a_1^{n-m}\right) \Rightarrow x_1^m = y_1^m$$
$$[A\left(a_1^{n-m}, x_1^m\right) = A\left(a_1^{n-m}, y_1^m\right) \Rightarrow x_1^m = y_1^m].$$

Then (Q, A) is an (m,n)-semigroup.

Sketch of the proof.

$$A\left(a_1^{i-1}, A\left(a_i^{i+n-1}\right), a_{i+n}^{2n-m}\right) = A\left(a_1^i, A(a_{i+1}^{i+n}), a_{i+n+1}^{2n-m}\right) \Rightarrow$$
$$A\left(b_1, A\left(a_1^{i-1}, A\left(a_i^{i+n-1}\right), a_{i+n}^{2n-m}\right), b_2^{n-m}\right) =$$
$$A\left(b_1, A\left(a_1^i, A\left(a_{i+1}^{i+n}\right), a_{i+n+1}^{2n-m}\right), b_2^{n-m}\right) \Rightarrow$$
$$A\left(A\left(b_1, a_1^{i-1}, A\left(a_i^{i+n-1}\right), a_{i+n}^{2n-m-1}\right), a_{2n-m}, b_2^{n-m}\right) =$$
$$A\left(A\left(b_1, a_1^i, A\left(a_{i+1}^{i+n}\right), a_{i+n+1}^{2n-m-1}\right), a_{2n-m}, b_2^{n-m}\right) \Rightarrow$$

$$A\left(b_1, a_1^{i-1}, A\left(a_i^{i+n-1}\right), a_{i+n}^{2n-m-1}\right) = A\left(b_1, a_1^i, A\left(a_{i+1}^{i+n}\right), a_{i+n+1}^{2n-m-1}\right).$$

(See, also 3.5 in [7].) □

3. Main results

3.1. Theorem: *Let $2m \le n < 3m$ and let (Q, A) be an (n, m)-groupoid $(n, m \in N)$. Then, (Q, A) is an (n, m)-group iff there are mappings $^{-1}$ and* **e** *respectively of the sets Q^{n-m} and Q^{n-2m} into the set Q^m such that the laws*

(1L) $$A\left(A\left(x_1^n\right), x_{n+1}^{2n-m}\right) = A\left(x_1, A\left(x_2^{n+1}\right), x_{n+2}^{2n-m}\right),$$

(1) $$A\left(A\left(x_1^n\right), x_{n+1}^{2n-m}\right) = A\left(x_1^{n-m}, A\left(x_{n-m+1}^{2n-m}\right)\right),$$

(2R) $$A\left(x_1^m, a_1^{n-2m}, \mathbf{e}\left(a_1^{n-2m}\right)\right) = x_1^m \quad \textit{and}$$

(3R) $$A\left(x_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) = \mathbf{e}\left(a_1^{n-2m}\right)$$

hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$.

Remark: For $m = 1:\ n = 2$ and (1L)=(1). See, also 3.3.

Proof. *a)* $\Rightarrow$:

Let (Q, A) be an (n, m)-group. Then, by Proposition 2.1, there is an algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the type $< (n, m), (n - m, m), (n - 2m, m) >$ in which the laws (1L),(1),(2R) and (3R).

b) $\Leftarrow$: Let $Q, \{A, ^{-1}, \mathbf{e}\})$ be an algebra of the type $< (n, m), (n - m, m), (n - 2m, m) >$ in which the laws (1L),(1),(2R) and (3R) are satisfied. Then the following statements hold:

°1 For every $x_1^m, y_1^m \in Q^m$ and for every sequence a_1^{n-m} over Q the following implication holds
$$A\left(x_1^m, a_1^{n-m}\right) = A\left(y_1^m, a_1^{n-m}\right) \Rightarrow x_1^m = y_1^m.$$

°2 (Q, A) is an (n, m)-semigroup.

°3 The law (2L) from 2 holds in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$.

°4 The law (3L) from 2 holds in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$.

°5 For every $x_1^m, y_1^m \in Q^m$ and for every sequence a_1^{n-m} over Q the following implication holds
$$A\left(a_1^{n-m}, x_1^m\right) = A\left(a_1^{n-m}, y_1^m\right) \Rightarrow x_1^m = y_1^m.$$

°6 For every $x_1^m, y_1^m, b_1^m, c_1^m \in Q^m$ and for every sequence a_1^{n-2m} over Q the following equivalences hold
$$A\left(x_1^m, a_1^{n-2m}, b_1^m\right) = c_1^m \Leftrightarrow x_1^m = A\left(c_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right) \quad \text{and}$$
$$A\left(b_1^m, a_1^{n-2m}, y_1^m\right) = c_1^m \Leftrightarrow y_1^m = A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, c_1^m\right).$$

The proof of the statement °1 :

By the assumption that the laws (1),(2R) and (3R) hold in $(Q, \{A, ^{-1}, \mathbf{e}\})$, we have that for every $b_1^m, x_1^m, y_1^m \in Q^m$ and for every sequence a_1^{n-2m} over Q the following series of implications hold

$$A\left(x_1^m, a_1^{n-2m}, b_1^m\right) = A\left(y_1^m, a_1^{n-2m}, b_1^m\right) \Rightarrow$$
$$A\left(A\left(x_1^m, a_1^{n-2m}, b_1^m\right), a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right) =$$
$$A\left(A\left(y_1^m, a_1^{n-2m}, b_1^m\right), a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right) \Rightarrow$$
$$A\left(x_1^m, a_1^{n-2m}, A\left(b_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right)\right) =$$
$$A\left(y_1^m, a_1^{n-2m}, A\left(b_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right)\right) \Rightarrow$$
$$A\left(x_1^m, a_1^{n-2m}, \mathbf{e}\left(a_1^{n-2m}\right)\right) = A\left(y_1^m, a_1^{n-2m}, \mathbf{e}\left(a_1^{n-2m}\right)\right) \Rightarrow$$
$$x_1^m = y_1^m.$$

The proof of the statement °2 :

For $m = 1 (n = 2 = m+1)$ the statement °2 is an immediate consequence of the definition of a semigroup and of the assumption that the law (1L) $/{=}(1)/$ holds in (Q, A). For $m > 1 (: n \geq 2m > m+1)$ the statement °2 holds by the assumption that the law (1L) holds in (Q, A), by statement °1 and by Proposition 2.2.

The proof of the statement °3 :

By the assumption the laws (2R) and (3R) hold in $(Q, \{A, ^{-1}, \mathbf{e}\})$, and also by °1 and °2, we conclude that for every $x_1^m, y_1^m = Q^m$ and for every sequence a_1^{n-2m} over Q the following sequence of implications holds:

$$A\left(\mathbf{e}\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right) = y_1^m \Rightarrow$$
$$A\left(A\left(\mathbf{e}\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right), a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) =$$
$$A\left(y_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) \Rightarrow$$
$$A\left(\mathbf{e}\left(a_1^{n-2m}\right), a_1^{n-2m}, A\left(x_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right)\right) =$$
$$A\left(\left(y_1^m, a_1^{n-2m}\right), \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) \Rightarrow A\left(\mathbf{e}\left(a_1^{n-2m}\right), a_1^{n-2m}, \mathbf{e}\left(a_1^{n-2m}\right)\right) =$$
$$A\left(y_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) \Rightarrow$$
$$\mathbf{e}\left(a_1^{n-2m}\right) = A\left(y_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) \Rightarrow$$
$$A\left(x_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) = A\left(y_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) \Rightarrow$$

$$x_1^m = y_1^m.$$

The proof of the statement $^\circ 4$:

By the assumption the laws (2R) and (3R) hold in $(Q, \{A, ^{-1}, \mathbf{e}\})$, and also by $^\circ 1 - ^\circ 3$, we conclude that for every $x_1^m, y_1^m \in Q^m$ and for every sequence a_1^{n-2m} over Q the following sequence of implications holds:

$$A\left(\left(a_1^{n-2m}, x_1^m\right)^{-1}, a_1^{n-2m}, x_1^m\right) = y_1^m \Rightarrow$$
$$A\left(A\left(\left(a_1^{n-2m}, x_1^m\right)^{-1}, a_1^{n-2m}, x_1^m\right), a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) =$$
$$A\left(y_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) \Rightarrow$$
$$A\left(\left(a_1^{n-2m}, x_1^m\right)^{-1}, a_1^{n-2m}, A\left(x_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right)\right) =$$
$$A\left(y_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) \Rightarrow A\left(\left(a_1^{n-2m}, x_1^m\right)^{-1}, a_1^{n-2m}, \mathbf{e}\left(a_1^{n-2m}\right)\right) =$$
$$A\left(y_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) \Rightarrow A\left(\mathbf{e}\left(a_1^{n-2m}\right), a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) =$$
$$A\left(y_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) \Rightarrow y_1^m = \mathbf{e}\left(a_1^{n-2m}\right).$$

The proof of the statement $^\circ 5$:

By $^\circ 2$ and by $^\circ 3$ – $^\circ 4$, we conclude that for every $x_1^m, y_1^m, b_1^m \in Q^m$ and for every sequence a_1^{n-2m} over Q the following series of implications holds:

$$A\left(b_1^m, a_1^{n-2m}, x_1^m\right) = A\left(b_1^m, a_1^{n-2m}, y_1^m\right) \Rightarrow$$
$$A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, A\left(b_1^m, a_1^{n-2m}, x_1^m\right)\right) =$$
$$A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, A\left(b_1^m, a_1^{n-2m}, y_1^m\right)\right) \Rightarrow$$
$$A\left(A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, b_1^m\right), a_1^{n-2m}, x_1^m\right) =$$
$$A\left(A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, b_1^m\right), a_1^{n-2m}, y_1^m\right) \Rightarrow$$
$$A\left(\mathbf{e}\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right) = A\left(\mathbf{e}\left(a_1^{n-2m}\right), a_1^{n-2m}, y_1^m\right) \Rightarrow$$
$$x_1^m = y_1^m.$$

The sketch of the proof of $^\circ 6$:

$$A\left(x_1^m, a_1^{n-2m}, b_1^m\right) = c_1^m \Leftrightarrow$$
$$A\left(A\left(x_1^m, a_1^{n-2m}, b_1^m\right), a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right) =$$
$$A\left(c_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right) \Leftrightarrow$$
$$A\left(x_1^m, a_1^{n-2m}, A\left(b_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right)\right) =$$

$$A\left(c_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right) \Leftrightarrow$$
$$A\left(x_1^m, a_1^{n-2m}, \mathbf{e}\left(a_1^{n-2m}\right)\right) = A\left(c_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right) \Leftrightarrow$$
$$x_1^m = A\left(c_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right)$$

/:°1,° 2,(2R),(3R)/.

Similarly:

$$A\left(b_1^m, a_1^{n-2m}, y_1^m\right) = c_1^m \Leftrightarrow$$
$$A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, A\left(b_1^m, a_1^{n-2m}, y_1^m\right)\right) =$$
$$A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, c_1^m\right) \Leftrightarrow \cdots \Leftrightarrow$$
$$y_1^m = A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, c_1^m\right)$$

/:°5,° 2,° 3,° 4,]. □

3.2. Theorem: *Let $n \geq 3m$ and let (Q, A) be an (n, m)-groupoid $(n, m \in N)$. Then, (Q, A) is an (n, m)-group iff there are mappings $^{-1}$ and* $\mathbf{e}$ *respectively of the sets Q^{n-m} and Q^{n-2m} into the set Q^m such that the laws*

(1L) $$A\left(A\left(x_1^n\right), x_{n+1}^{2n-m}\right) = A\left(x_1, A\left(x_2^{n+1}\right), x_{n+2}^{2n-m}\right),$$

(1Lm) $$A\left(A\left(a_1^m, b_1^{n-m}\right), c_1^m, d_1^{n-2m}\right) = A\left(a_1^m, A\left(b_1^{n-m}, c_1^m\right), d_1^{n-2m}\right),$$

(2R) $$A\left(x_1^m, a_1^{n-2m}, \mathbf{e}\left(a_1^{n-2m}\right)\right) = x_1^m \quad \text{and}$$

(3R) $$A\left(x_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) = \mathbf{e}\left(a_1^{n-2m}\right)$$

hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$.

Remark: For $m = 1$: (1Lm)=(1L). See, also 3.3.

Proof. *a)* $\Rightarrow$:

Let (Q, A) be an (n, m)-group. Then, by Proposition 2.1, there is an algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the type $< (n, m), (n-m, m), (n-2m, m) >$ in which the laws (1L), (1Lm), (2R) and (3R) hold.

b) $\Leftarrow$: Let $(Q, \{A, ^{-1}, \mathbf{e}\})$ be an algebra of the type $< (n, m), (n-m, m), (n-2m, m) >$ in which the laws (1L), (1Lm), (2R) and (3R) are satisfied. Then the statenments °1 −° 6 from the proof of Theorem 3.1 hold.

The sketch of the proof of °1 :

$$A\left(x_1^m, b_1^m, a_1^{n-2m}\right) = A\left(y_1^m, b_1^m, a_1^{n-2m}\right) \Rightarrow$$
$$A\left(A\left(x_1^m, b_1^m, a_1^{n-2m}\right), \mathbf{e}\left(a_1^{n-2m}\right), c_1^{n-3m}, \mathbf{e}\left(b_1^m, c_1^{n-3m}\right)\right) =$$
$$A\left(A\left(y_1^m, b_1^m, a_1^{n-2m}\right), \mathbf{e}\left(a_1^{n-2m}\right), c_1^{n-3m}, \mathbf{e}\left(b_1^m, c_1^{n-3m}\right)\right)\right) \Rightarrow$$

$$A\left(x_1^m, A\left(b_1^m, a_1^{n-2m}, \mathbf{e}\left(a_1^{n-2m}\right)\right), c_1^{n-3m}, \mathbf{e}\left(b_1^m, c_1^{n-3m}\right)\right) =$$
$$A\left(y_1^m, A\left(b_1^m, a_1^{n-2m}, \mathbf{e}\left(a_1^{n-2m}\right)\right), c_1^{n-3m}, \mathbf{e}\left(b_1^m, c_1^{n-3m}\right)\right) \Rightarrow$$
$$A\left(x_1^m, b_1^m, c_1^{n-3m}, \mathbf{e}\left(b_1^m, c_1^{n-3m}\right)\right) =$$
$$A\left(y_1^m, b_1^m, c_1^{n-3m}, \mathbf{e}\left(b_1^m, c_1^{n-3m}\right)\right) \Rightarrow x_1^m = y_1^m$$

/: (1Lm),(2R)/.

The proof of the statement °2 :

The statement °2 holds by the assumption that the law (1L) holds in (Q,A), by statement °1 and by Proposition 2.2; $n > m+1(: n \geq 3m)$.

For proofs of the statements °3 –° 6 see the Proof of Theorem 3.1. □

3.3. Remark: *For $n \geq 2m$ and $m = 1$ $(: n \geq 2)$ the following proposition holds [7]: Let $n \geq 2$ and let (Q,A) be an n-groupoid. Then, (Q,A) is an n-group iff there are mappings $^{-1}$ and* $\mathbf{e}$ *respectively of the sets Q^{n-1} and Q^{n-2} into the set Q such that the laws*

(1L) $$A\left(A\left(x_1^n\right), x_{n+1}^{2n-1}\right) = A\left(x_1, A\left(x_2^{n+1}\right), x_{n+2}^{2n-m}\right),$$

(2R) $$A\left(x, a_1^{n-2}, \mathbf{e}\left(a_1^{n-2}\right)\right) = x \quad \text{and}$$

(3R) $$A\left(x, a_1^{n-2}, \left(a_1^{n-2}, x\right)^{-1}\right) = \mathbf{e}\left(a_1^{n-2}\right)$$

hold in the algebra $(Q, \{A,^{-1}, \mathbf{e}\})$. In addition: The laws (1L),(2R) and (3R) are independent. □

3.4. Theorem: *Let $n \geq 2m$ and let (Q,A) be an (n,m)-groupoid. Then, (Q,A) is an (n,m)-group iff there are mappings $^{-1}$ and* $\mathbf{e}$ *respectively of the sets Q^{n-m} and Q^{n-2m} into the set Q^m such that the laws*

(1L) $$A\left(A\left(x_1^n\right), x_{n+1}^{2n-m}\right) = A\left(x_1, A\left(x_2^{n+1}\right), x_{n+2}^{2n-m}\right),$$

(2L) $$A\left(\mathbf{e}\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right) = x_1^m \quad \text{and}$$

(4R) $$A\left(A\left(x_1^m, a_1^{n-2m}, b_1^m\right), a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right) -1\right) = x_1^m$$

hold in the algebra $(Q, \{A,^{-1}, \mathbf{e}\})$.

Proof. *a)* $\Rightarrow$:

Let (Q,A) be an (n,m)-group. Then, by Proposition 2.1, there is an algebra $(Q, \{A,^{-1}, \mathbf{e}\})$ of the type $< (n,m), (n-m,m), (n-2m,m) >$ in which the laws (1L), (2L) and (4R) hold.

b) $\Leftarrow$: Let $(Q, \{A,^{-1}, \mathbf{e}\})$ be an algebra of the type $< (n,m), (n-m,m), (n-2m) >$ in which the laws (1L), (2L) and (4R) are satisfied. Then the following statements hold:

$\overline{1}$ For every $x_1^m, y_1^m \in Q^m$ and for every sequence a_1^{n-m} over Q the following implication holds

$$A\left(x_1^m, a_1^{n-m}\right) = A\left(y_1^m, a_1^{n-m}\right) \Rightarrow x_1^m = y_1^m.$$

$\overline{2}$ (Q, A) is an (n, m)-semigroup.

$\overline{3}$ The laws (3R),(2R) and (3L) from 2 hold in the algebra $(Q, \{A,^{-1}, \mathbf{e}\})$.

$\overline{4}$ For every $x_1^m, y_1^m \in Q^m$ and for every sequence a_1^{n-m} over Q the following implication holds

$$A\left(a_1^{n-m}, x_1^m\right) = A\left(a_1^{n-m}, y_1^m\right) \Rightarrow x_1^m = y_1^m.$$

$\overline{5}$ For every $x_1^m, y_1^m, b_1^m, c_1^m \in Q^m$ and for every sequence a_1^{n-2m} over Q the following equivalences hold

$$A\left(x_1^m, a_1^{n-2m}, b_1^m\right) = c_1^m \Leftrightarrow x_1^m = A\left(c_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right) \quad \text{and}$$

$$A\left(b_1^m, a_1^{n-2m}, y_1^m\right) = c_1^m \Leftrightarrow y_1^m = A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, c_1^m\right).$$

The sketch of the proof $\overline{1}$:

$$A\left(x_1^m, a_1^{n-2m}, b_1^m\right) = A\left(y_1^m, a_1^{n-2m}, b_1^m\right) \Rightarrow$$
$$A\left(A\left(x_1^m, a_1^{n-2m}, b_1^m\right), a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right) =$$
$$A\left(A\left(y_1^m, a_1^{n-2m}, b_1^m\right), a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right) \Rightarrow$$
$$x_1^m = y_1^m$$

/:(4R)/.

The proof of the statement $\overline{2}$:

For $n = 2m$ and $m = 1$ the statement $\overline{2}$ is an immediate consequence of the definition of a semigroup and of the assumption that the law (1L) holds in (Q, A). For $n \geq 2m > m+1$ the statement $\overline{2}$ holds by the assumption that the law (1L) holds in (Q, A), by stement $\overline{1}$ and by Proposition 2.2.

The sketch of the proof $\overline{3}$:

a) $$A\left(A\left(\mathbf{e}\left(a_1^{n-2m}\right), a_1^{n-2m}, b_1^m\right), a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right) = \mathbf{e}\left(a_1^{n-2m}\right) \Rightarrow$$
$$A\left(b_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right) = \mathbf{e}\left(a_1^{n-2m}\right)$$

/:(2L),(4R),$x_1^m = \mathbf{e}(a_1^{n-2m})$/.

b) $$A\left(x_1^m, a_1^{n-2m}, \mathbf{e}\left(a_1^{n-2m}\right)\right) = y_1^m \Rightarrow$$
$$A\left(A\left(x_1^m, a_1^{n-2m}, \mathbf{e}\left(a_1^{n-2m}\right)\right), a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) =$$

$$A\left(y_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) \Rightarrow$$
$$A\left(x_1^m, a_1^{n-2m}, A\left(\mathbf{e}\left(a_1^{n-2m}\right), a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right)\right) =$$
$$A\left(y_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) \Rightarrow$$
$$A\left(x_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) = A\left(y_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) \Rightarrow$$
$$x_1^m = y_1^m$$

/: $\overline{2}$,(2L), $\overline{1}$ /.

c)
$$A\left(\left(a_1^{n-2m}, x_1^m\right)^{-1}, a_1^{n-2m}, x_1^m\right) = y_1^m \Rightarrow$$
$$A\left(A\left(a_1^{n-2m}, x_1^m\right)^{-1}, \left(a_1^{n-2m}, x_1^m\right), a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) =$$
$$A\left(y_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) \Rightarrow$$
$$A\left(\left(a_1^{n-2m}, x_1^m\right)^{-1}, a_1^{n-2m}, A\left(x_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right)\right) =$$
$$A\left(y_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) \Rightarrow$$
$$A\left(\left(a_1^{n-2m}, x_1^m\right)^{-1}, a_1^{n-2m}, \mathbf{e}\left(a_1^{n-2m}\right)\right) =$$
$$A\left(y_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) \Rightarrow$$
$$A\left(\mathbf{e}\left(a_1^{n-2m}\right), a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) =$$
$$A\left(y_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, x_1^m\right)^{-1}\right) \Rightarrow \mathbf{e}\left(a_1^{n-2}\right) = y_1^m$$

/: $\overline{2}$,(3R)–a),(2L),(2R)–b), $\overline{1}$ /.

The sketch of the proof $\overline{4}$:

$$A\left(b_1^m, a_1^{n-2m}, x_1^m\right) = A\left(b_1^m, a_1^{n-2m}, y_1^m\right) \Rightarrow$$
$$A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, A\left(b_1^m, a_1^{n-2m}, x_1^m\right)\right) =$$
$$A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, A\left(b_1^m, a_1^{n-2m}, y_1^m\right)\right) \Rightarrow$$
$$A\left(A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, b_1^m\right), a_1^{n-2m}, x_1^m\right) =$$
$$A\left(A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, b_1^m\right), a_1^{n-2m}, y_1^m\right) \Rightarrow$$
$$A\left(\mathbf{e}\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right) = A\left(\mathbf{e}\left(a_1^{n-2m}\right), a_1^{n-2m}, y_1^m\right) \Rightarrow$$
$$x_1^m = y_1^m$$

/: $\overline{2}$,(3L)–c),(2L) /.

The sketch of the proof $\overline{5}$:

$$A\left(b_1^m, a_1^{n-2m}, y_1^m\right) = c_1^m \Leftrightarrow$$
$$A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, A\left(b_1^m, a_1^{n-2m}, y_1^m\right)\right) =$$
$$A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, c_1^m\right) \Leftrightarrow$$
$$A\left(A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, b_1^m\right), a_1^{n-2m}, y_1^m\right) =$$
$$A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, c_1^m\right) \Leftrightarrow$$
$$A\left(\mathbf{e}\left(a_1^{n-2m}\right), a_1^{n-2m}, y_1^m\right) = A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, c_1^m\right) \Leftrightarrow$$
$$y_1^m = A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, c_1^m\right)$$

/: $\overline{4}, \overline{2}$, (3L)-c),(2L) /.

$$A\left(x_1^m, a_1^{n-2m}, b_1^m\right) = c_1^m \Leftrightarrow$$
$$A\left(A\left(x_1^m, a_1^{n-2m}, b_1^m\right), a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right) =$$
$$A\left(c_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right) \Leftrightarrow$$
$$x_1^m = A\left(c_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right)$$

/: $\overline{1}, (4R)$ /. □

Similarly, it is possible to prove that the following proposition holds:

3.5. Theorem: *Let $n \geq 2m$ and let (Q, A) be an (n, m)-groupoid. Then, (Q, A) is an (n, m)-group iff there is a mapping $^{-1}$ of the set Q^{n-m} into the set Q^m such that the laws (1L),[or (1R)], (4L) and (4R) from 2 hold in the $(Q, \{A, ^{-1}\})$.*

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